How can we apply limit laws in conjunction with algebraic techniques to evaluate the exact value of limits?

Quick Check

Draw labeled sketches to illustrate and explain your reasoning for each question below.

1. If $f(3)=5$, what can you conclude about the limit of $f(x)$ as $x \to 3$.

2. If $\lim_{x \to 3} f(x)=5$, what can you conclude about $f(3)$?

Find and compare the limits.

1. $\lim_{x \to 2} (-x^2 + 5x - 2)$

2. $\lim_{x \to 1} \dfrac{x^3 + 4x + 1}{x^2 + 4}$

3. $\lim_{x \to 0} \dfrac{2}{x^2}$

4. $\lim_{x \to 1} \dfrac{x^2 - 1}{x - 1}$

🤔 What did the Direct Substitution Property tell us about evaluating limits?

Dividing Out Technique

⚠️ Notice the **indeterminate form**.

$$\lim_{x \to -3} \frac{x^2 + x - 6}{x + 3}$$

$$\lim_{x \to -1} \frac{x + 1}{x^2 - 1}$$

Rationalizing Technique

⚠️ Notice the **indeterminate form**.

$$\lim_{x \to 0} \frac{\sqrt{x+1} - 1}{x}$$

$$\lim_{x \to 4} \frac{2 - \sqrt{x}}{x - 4}$$

Practice: Find each limit, if it exists.

1. $\lim_{x \to 1} (3x^3 - 2x^2 + 1)$

2. $\lim_{x \to 3} \dfrac{2x + 4}{x + 3}$

3. $\lim_{x \to 2} \dfrac{\sqrt{x + 1}}{x - 4}$

4. $\lim_{x \to 0} (2x - 1)^{99}$

5. $\lim_{x \to -2} \dfrac{2 - x}{x^2 - 4}$

6. $\lim_{x \to 4} \dfrac{x^2 - 5x + 4}{x^2 - 2x - 8}$

7. $\lim_{x \to 0} \dfrac{\sqrt{2 + x} - \sqrt{2}}{x}$

8. $\lim_{x \to 0} \dfrac{\dfrac{1}{x + 4} - \dfrac{1}{4}}{x}$

Trigonometric limits use direct substituion on their domain.

Let a be a real number in the domain of the given trigonometric function.

1. $\lim_{x \to a} \sin(x) = \sin(a)$
2. $\lim_{x \to a} \cos(x) = \cos(a)$
3. $\lim_{x \to a} \tan(x) = \tan(a)$
4. $\lim_{x \to a} \cot(x) = \cot(a)$
5. $\lim_{x \to a} \sec(x) = \sec(a)$
6. $\lim_{x \to a} \csc(x) = \csc(a)$

Evaluate the following limits.

1. $\lim_{x \to 0} \tan(x)$
2. $\lim_{x \to \pi} x \cdot \cos(x)$
3. $\lim_{x \to 0} \cos^2(x)$

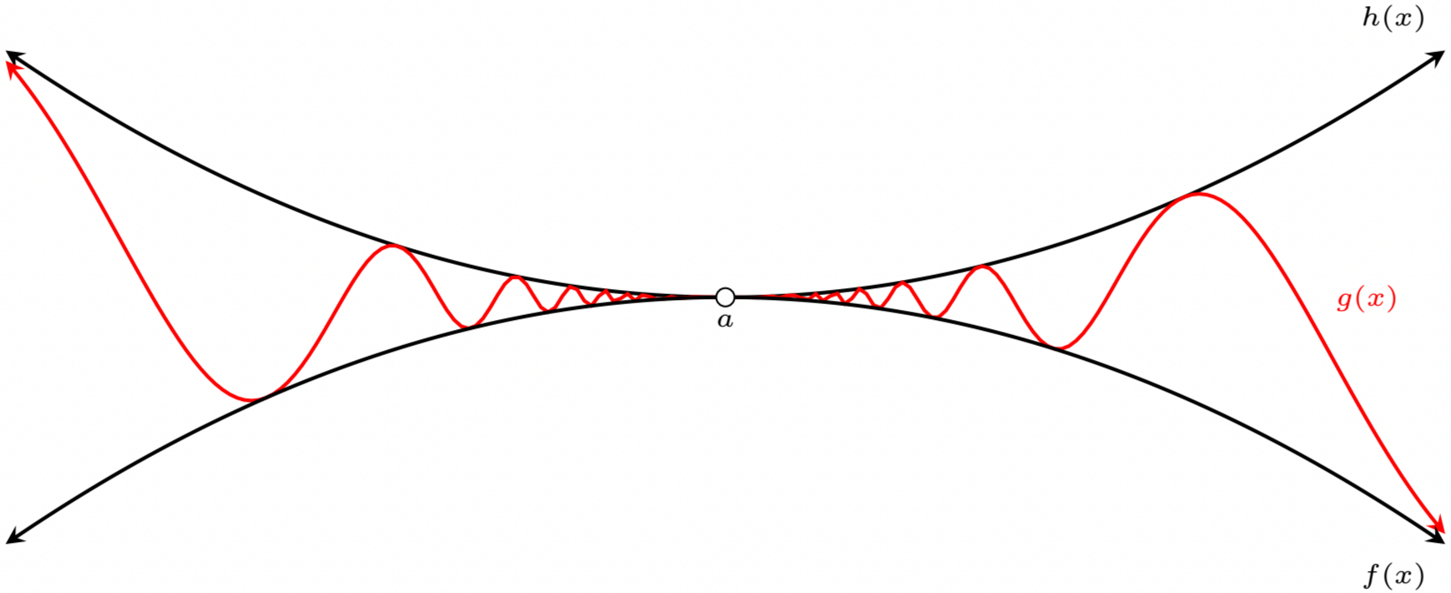
The Squeeze Theorem

If $f(x) \le g(x) \le h(x)$ when x is near a (except possibly at a) and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

then

$$\lim_{x \to a} g(x) = L$$



Find $\lim_{x\to 0} x^2 \cdot \sin(\frac{1}{x})$

~~$\lim_{x\to 0} x^2 \cdot \sin(\frac{1}{x}) = \lim_{x\to 0} x^2 \cdot \lim_{x\to 0} \sin(1/x)$~~

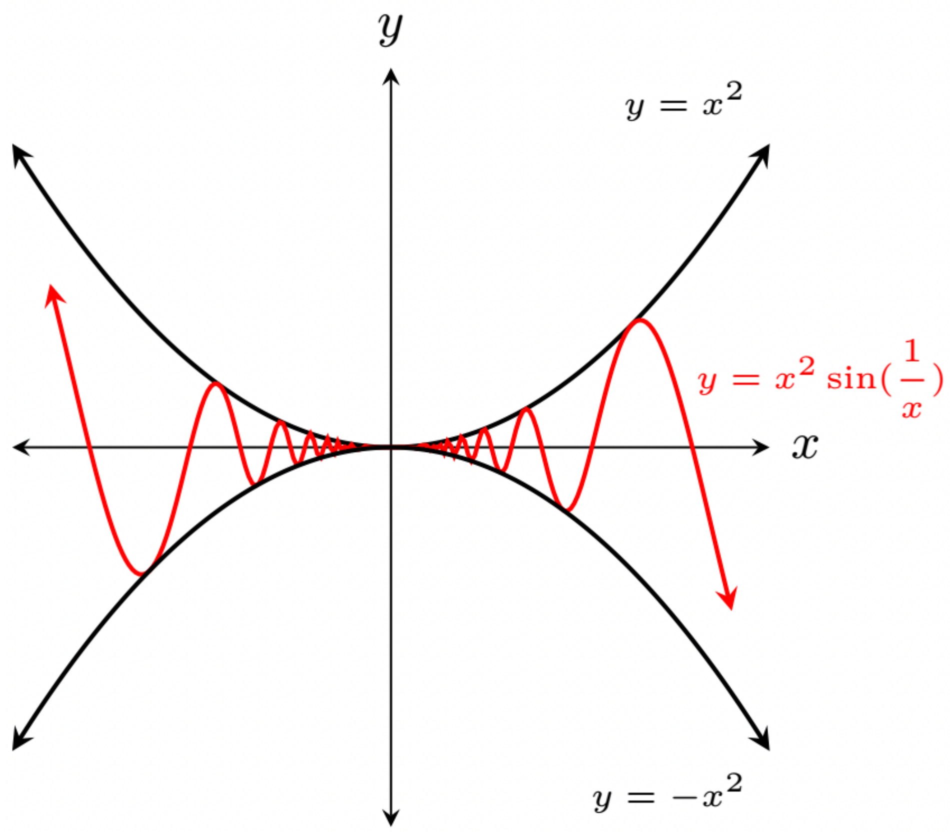
We cannot use limit laws because $\lim_{x\to 0} \sin(1/x)$ DNE.

Observe
$$-1 \le \sin(1/x) \le 1$$
$$-x^2 \le x^2 \cdot \sin(1/x) \le x^2$$

Since

$\lim_{x\to 0} x^2 = 0$ and $\lim_{x\to 0} -x^2 = 0$

By the Squeeze Theorem, $\lim_{x\to 0} x^2 \cdot \sin(\frac{1}{x}) = 0$



Special Trigonometric Limits

1. $\lim_{x \to 0} \frac{\sin x}{x} = 1$

2. $\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$

💭 Construct a graphical reasoning for (1). Try conjugate method to show second limit.

Find the following limits.

1. $\lim_{x \to 0} \frac{\tan x}{x}$

2. $\lim_{x \to 0} \frac{\sin 4x}{x}$

Determine the limit of each trigonometric function (if it exists).

1. $\lim_{x \to 0} \frac{\sin x}{3x}$
2. $\lim_{x \to 0} \frac{4 \cdot (1 - \cos x)}{x}$
3. $\lim_{\theta \to 0} \frac{\cos \theta \cdot \tan \theta}{\theta}$
4. $\lim_{x \to 0} \frac{\sin^2 x}{x}$
5. $\lim_{h \to 0} \frac{(1 - \cos h)^2}{h}$
6. $\lim_{x \to \pi/2} \frac{\cos x}{\cot x}$
7. $\lim_{x \to 0} \frac{\sin 2x}{\sin 3x}$
8. $\lim_{x \to 0} \frac{1 - \cos x}{\sin x}$