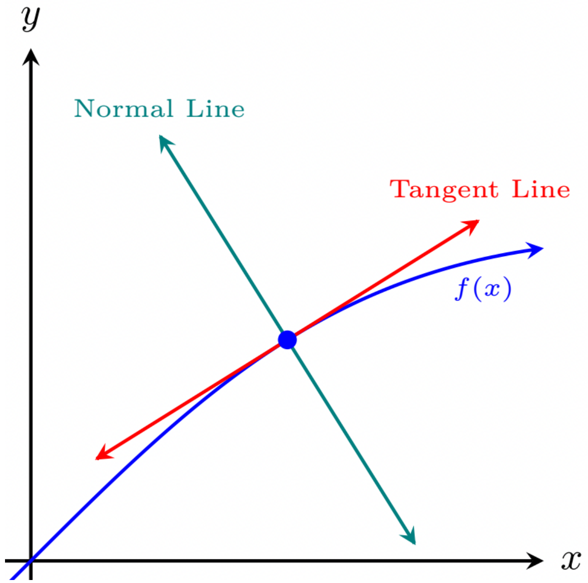
How do we find the derivative of a quotient of functions?

Quick Check

A Normal Line through $(a, f(a))$ is perpendicular to the tangent line through the same point on the curve of $f(x)$.



Write the equation of the Normal Line to the graph of $g(x) = 5 - x^2$ passing through the point $(1, 3)$.

Does the derivative of a quotient of functions $\frac{d}{dx}[\frac{f(x)}{g(x)}]$ work as expected?

Let $f(x) = 5x^3$ and $g(x) = x$.

1 Find the derivative of $\frac{f(x)}{g(x)}$ by first simplifying then taking the drivative of the result.

2 Check if the quotient of the derivatives works to achieve the same result or not.

$$\frac{d}{dx}[\frac{f(x)}{g(x)}] \stackrel{?}{=} \frac{\frac{d}{dx}[f(x)]}{\frac{d}{dx}[g(x)]}$$

✨ Simplification by Addition of Zero ➕0️⃣

Let $q(x) = \dfrac{f(x)}{g(x)}$, then by the definition of the derivative

$$q'(x) = \lim_{h \to 0} \frac{q(x+h) - q(x)}{h}$$

$$= \lim_{h \to 0} \frac{\dfrac{f(x+h)}{g(x+h)} - \dfrac{f(x)}{g(x)}}{h}$$

$$= \ldots$$

The Quotient Rule

The quotient of two differentiable functions, $\frac{f}{g}$, is itself differentiable at all values of x for which $g(x) \neq 0$.

$$\frac{d}{dx}[\frac{f(x)}{g(x)}] = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$$

🎤🎶 A yodel to remember

$$\frac{d}{dx}[\frac{high}{low}] = \frac{\text{lo}\ d\ (\text{hi}) - \text{hi}\ d\ (\text{lo})}{lo.lo}$$

Derivative of Quotients

Find the derivative using the quotient rule.

1 $f(x) = \dfrac{x}{1 + x^2}$

2 Find an equation for the tangent line to the curve $y = \dfrac{x^2 + 5}{2x}$ at the point $(5, 3)$. Check visually by graphing the function and the tangent line.

The General Power Rule

If n is a negative integer and $x \neq 0$, then

$$\frac{d}{dx}(x^n) = n \cdot x^{n-1}$$

If n is a negative integer, then $n = -m$, where m is a positive integer. It follows $x^n = x^{-m} = \frac{1}{x^m}$, and by quotient rule

$$\frac{d}{dx}(x^n) = \frac{d}{dx}\left(\frac{1}{x^m}\right) = \frac{0 - mx^{m-1}}{x^{2m}} = -mx^{-m-1} = nx^{n-1}$$

Although, we will prove this later, Power Rule extends to rational exponents as well.

General Power Rule and The Quotient Rule

1 Find the derivative using the Power Rule. Start by rewriting the expression.

$$g(x) = \frac{x^3 - 3x^2 + 4}{x^2}$$

2 Find the derivative using both the quotient rule and the general power rule.

$$y = \frac{5}{2x^3}$$

More Derivatives

Find the derivative using any method.

1. $y = \dfrac{1}{\sqrt[3]{x^2}}$

2. $g(x) = \dfrac{\sin x}{x^2}$

3. $f(x) = \dfrac{x^2 + 5x - 1}{x^2}$

4. $h(x) = \dfrac{3x^2 - 1}{2x + 5}$

5. Find the tangent to Newton's Serpentine, $y = \dfrac{4x}{x^2 + 1}$, at $(-1, -2)$, $(0, 0)$, and $(1, 2)$.

Notation for Higher-Order Derivatives

First Derivative	y'	$f'(x)$	$\dfrac{dy}{dx}$
Second Derivative	y''	$f''(x)$	$\dfrac{d^2y}{dx^2}$
Third Derivative	y'''	$f'''(x)$	$\dfrac{d^3y}{dx^3}$
Fourth Derivative	$y^{(4)}$	$f^{(4)}(x)$	$\dfrac{d^4y}{dx^4}$
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
n^{th} Derivative	$y^{(n)}$	$f^{(n)}(x)$	$\dfrac{d^ny}{dx^n}$

Finding Higher Order Derivatives

1. Find the first four derivatives of $y = x^4 - x^3 + 5x^2 + 1$.

2. $f'(x) = 3x^2$, What is $f''(x)$.

3. $f^{(3)}(x) = 10x^2$, find $f^{(6)}(x)$.

4. Find the second derivative of $y = \frac{x+1}{x}$.