# How are integrals related to area under a curve?

## Quick Check

1. Find approximate area under the graph of $f(x) = x^2$ from $x = 0$ to $x = 3$ using 3 rectangles. Explain whether your computed value is an over-estimate or an under-estimate.
2. Find the exact area of the region bounded by the graph of $f(x) = x^2$, $x = 0$, $x = 3$, and the $x-$axis using the limit definition.

# Riemann Sum

Let $f$ be defined on the closed interval $[a, b]$, and let $\Delta$ be a partition of $[a, b]$ given by

$$a = x_0 < x_1 < x_2 < \ . \ . \ . \ < x_{n-1} < x_n = b$$

where $\Delta x_i$ is the width of the $i$th subinterval

$$[x_{i-1}, x_i].$$

$i$th subinterval

If $c_i$ is *any* point in the $i$th subinterval, then the sum

$$\sum_{i=1}^{n} f(c_i)\,\Delta x_i, \quad x_{i-1} \le c_i \le x_i$$

is called a **Riemann sum** of $f$ for the partition $\Delta$.

# Definition of the Definite Integral

If $f$ is defined on the closed interval $[a, b]$ and the limit of Riemann sums over partitions $\Delta$

$$\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(c_i)\, \Delta x_i$$

exists (as described above), then $f$ is said to be **integrable** on $[a, b]$ and the limit is denoted by

$$\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(c_i)\, \Delta x_i = \int_a^b f(x)\, dx.$$

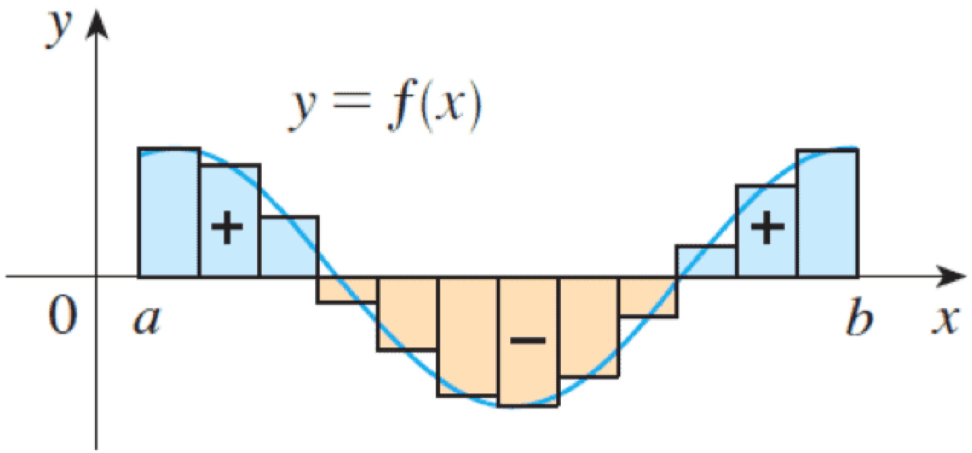
The limit is called the **definite integral** of $f$ from $a$ to $b$. The number $a$ is the **lower limit** of integration, and the number $b$ is the **upper limit** of integration.

# Evaluating a Definite Integral as a Limit

1. Draw a sketch of the region indicated by the integral below.
2. Use the definition to evaluate the value of the integral.

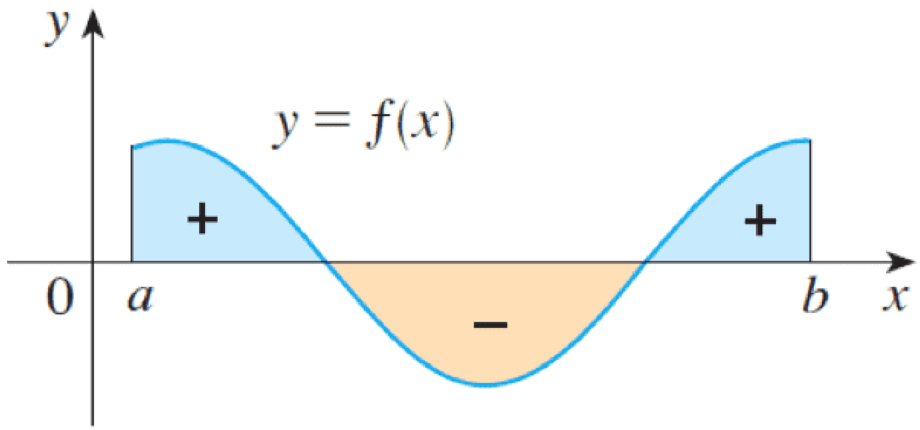
$$\int_{-2}^{1} 2x\,dx$$

# When does definite integral represent area?



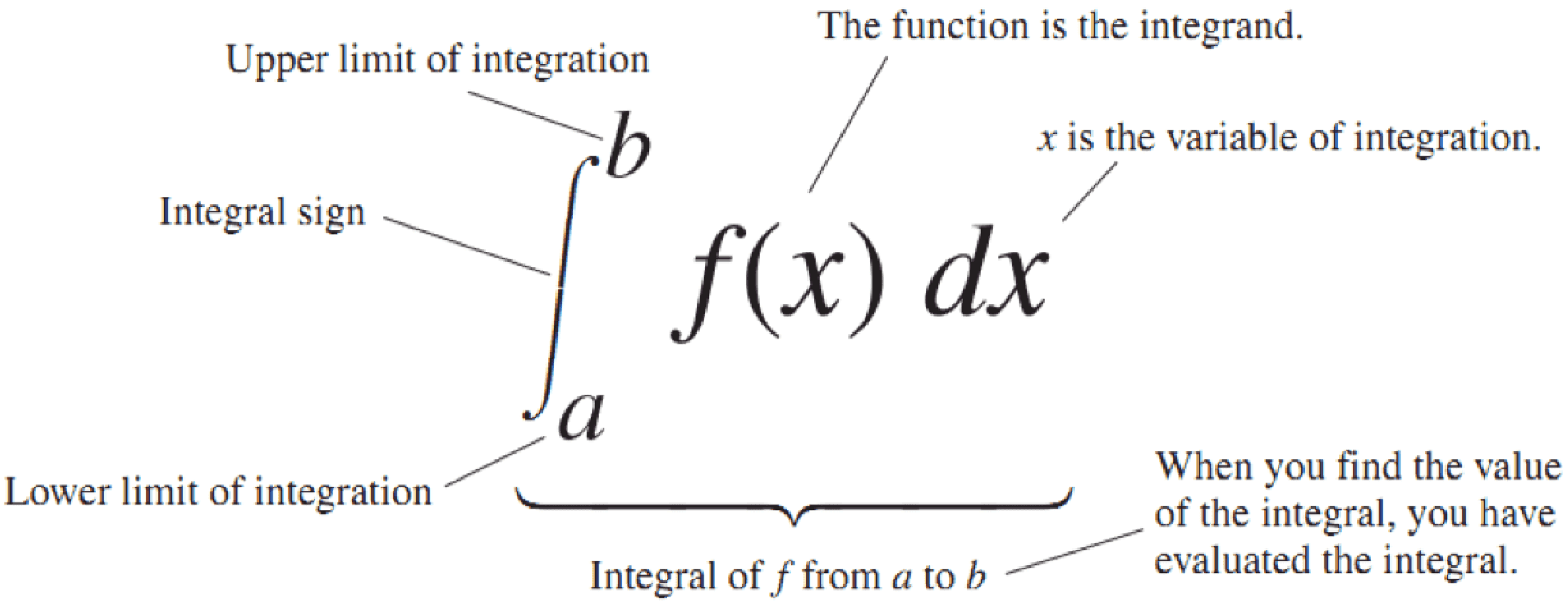

$\Sigma f(x_i^*)\,\Delta x$ is an approximation to the net area.

the difference of areas



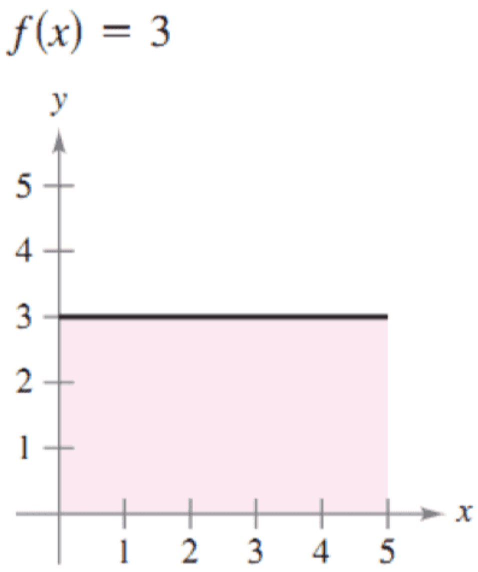

$\int_a^b f(x)\,dx$ is the net area.

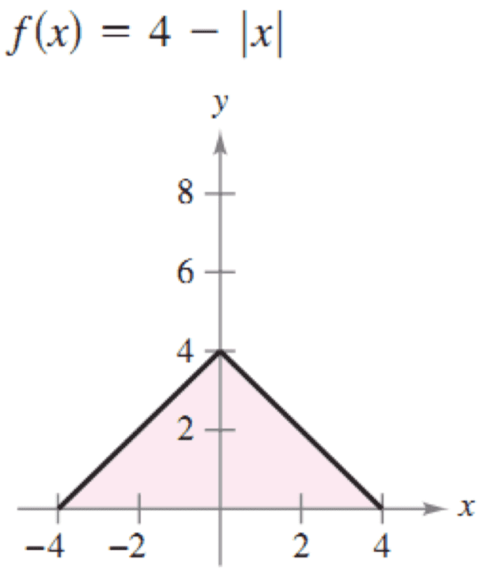
# The Definite Integral



# Set up the integral that yields the area of the region.

(Do not evaluate the integral.)



## Sketch & Evaluate

Sketch the region corresponding to each definite integral. Then evaluate each integral using a geometric formula.

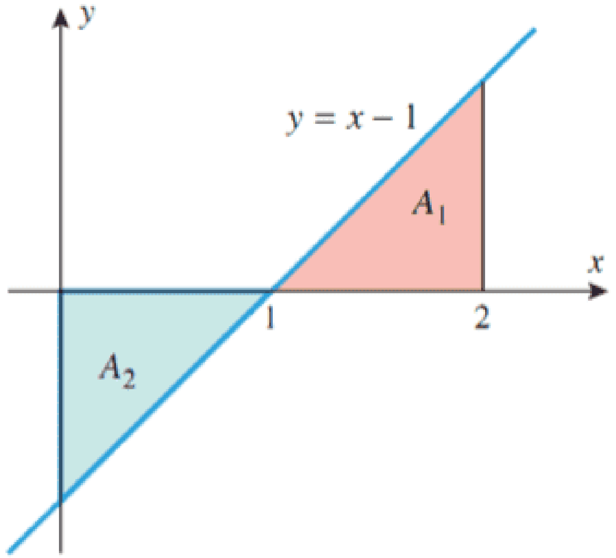
1. $\int_{1}^{3} 4\,dx$

2. $\int_{0}^{3} (x+2)\,dx$

3. $\int_{-2}^{2} \sqrt{4-x^2}\,dx$

## 🕸 'Net Area'

(a) $\int_0^2 (x-1)\,dx$  (b) $\int_0^1 (x-1)\,dx$




Write your answer in terms of $A_1$ and $A_2$ for each question.

# Two Special Definite Integrals

1. If $f$ is defined at $x = a$, then we define $\int_a^a f(x)\,dx = 0$.

2. If $f$ is integrable on $[a, b]$, then we define $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$.

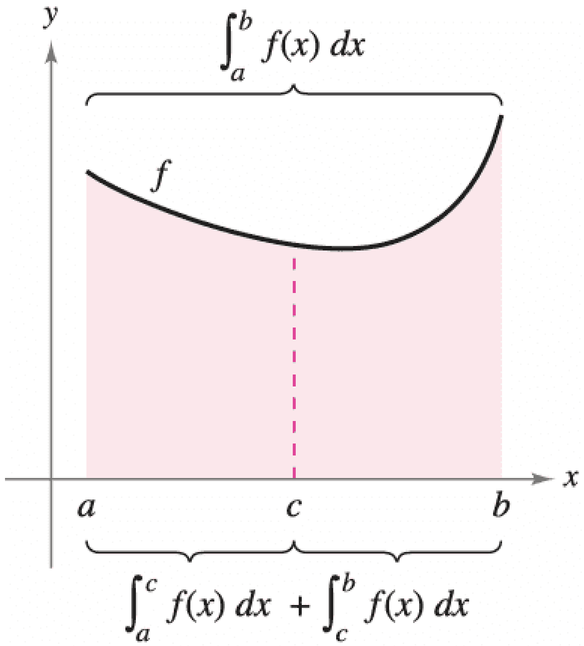
A $\int_{\pi}^{\pi} sin(x)\,dx$

B $\int_{3}^{0} x + 2\,dx$

## Additive Interval Property

If $f$ is integrable on the three closed intervals determined by $a$, $b$, and $c$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$



## Properties of Definite Integrals

If $f$ and $g$ are integrable on $[a, b]$ and $k$ is a constant, then the functions $kf$ and $f \pm g$ are integrable on $[a, b]$, and

1. $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$

2. $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx.$

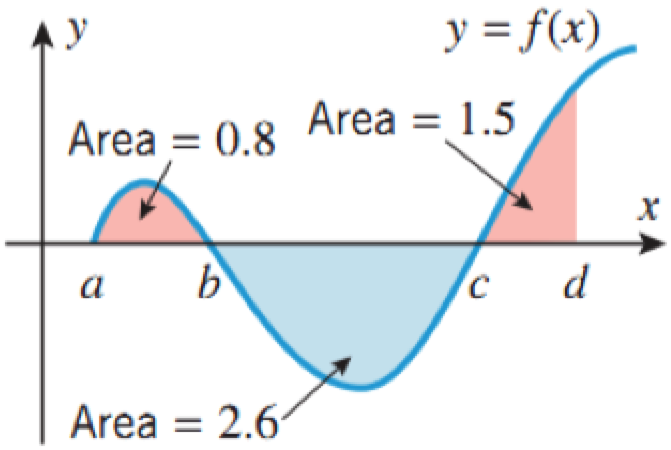
Find $\int_1^3 -x^2 + 4x - 3\,dx$ using $\int_1^3 x^2\,dx = \frac{26}{3}$ and $\int_1^3 x\,dx = 4$ and $\int_1^3 1\,dx = 2.$

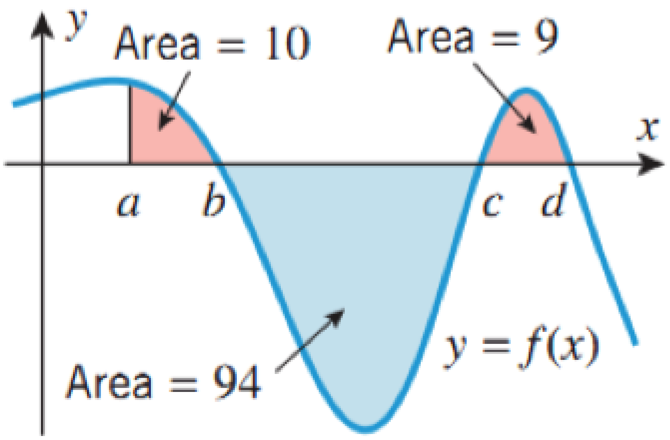
# Visual Understanding

Use the areas shown in the figure to find

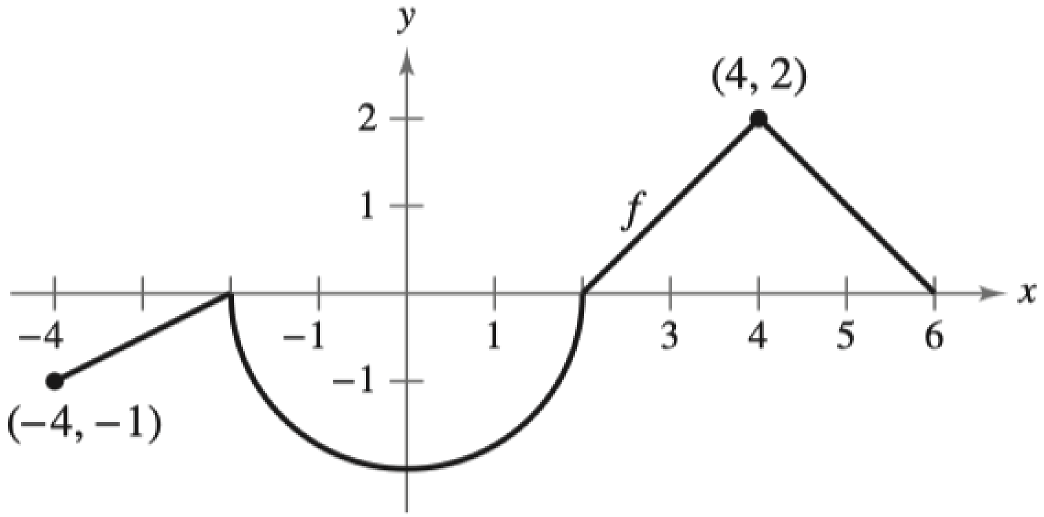
(a) $\int_a^b f(x)\,dx$ (b) $\int_b^c f(x)\,dx$

(c) $\int_a^c f(x)\,dx$ (d) $\int_a^d f(x)\,dx$.



# Visual Understanding




(a) $\int_{0}^{2} f(x)\,dx$

(b) $\int_{2}^{6} f(x)\,dx$

(c) $\int_{-4}^{2} f(x)\,dx$

(d) $\int_{-4}^{6} f(x)\,dx$

(e) $\int_{-4}^{6} |f(x)|\,dx$

(f) $\int_{-4}^{6} [f(x) + 2]\,dx$