How does the Fundamental Theorem help us explore the definition of the natural logarithm as an integral?

Quick Check

Sketch the region corresponding to each definite integral.
⚠️ DO NOT EVALUATE THE INTEGRAL.

1. $\displaystyle\int_0^5 (x+1)\,dx$

2. $\displaystyle\int_1^1 \frac{1}{t}\,dt$

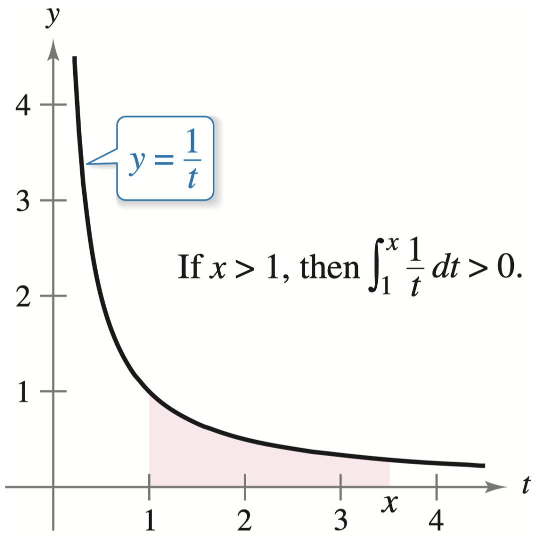
3. $\displaystyle\int_1^4 \frac{1}{t}\,dt$

Definition & Derivative

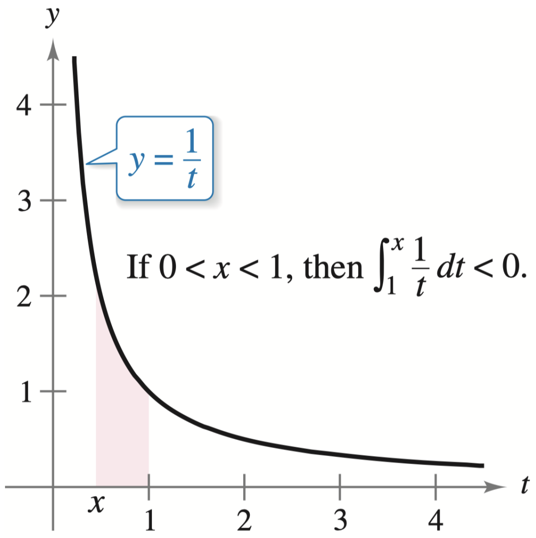
The **natural logarithmic function** is defined by

$$\ln(x) = \int_1^x \frac{1}{t}\,dt, \quad x > 0$$

The domain of $\ln(x)$ is the set of positive real numbers.



If $x > 1$, then $\ln x > 0$.

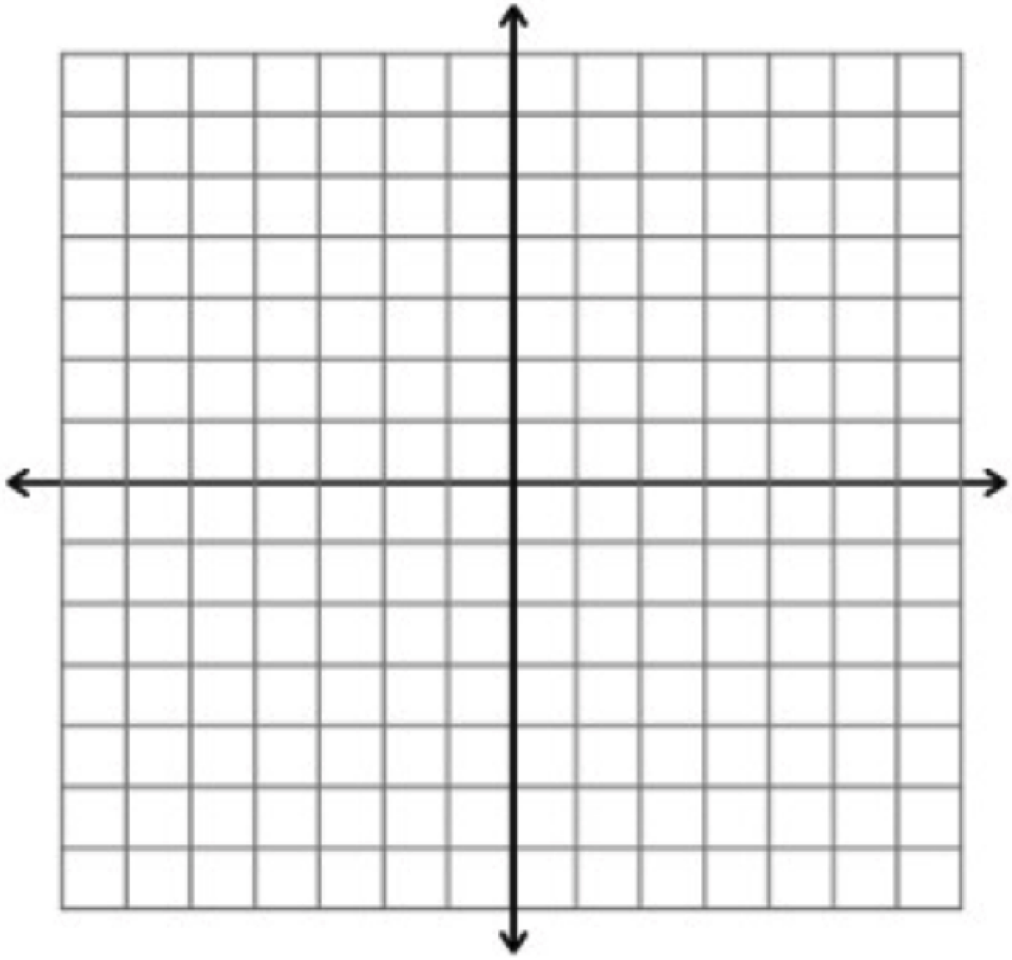


If $0 < x < 1$, then $\ln x < 0$.

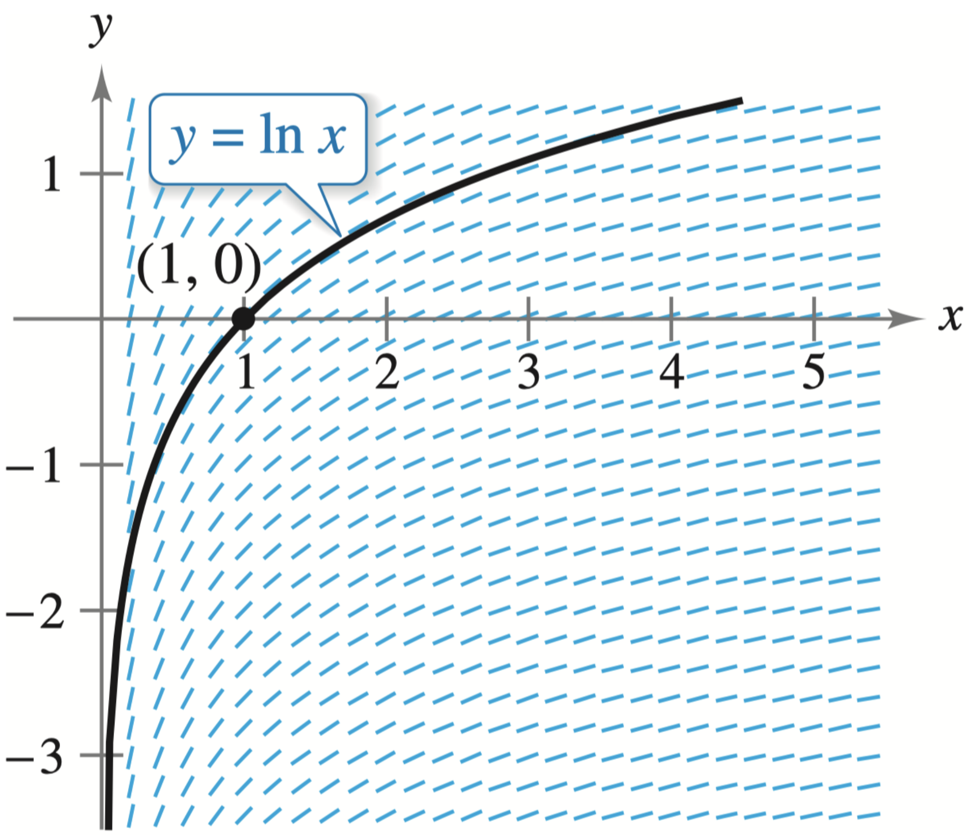
Graph of $\ln(x)$

To sketch the graph of $y = \ln(x)$, you can think of the natural logarithmic function as an antiderivative given by the differential equation

$$\frac{dy}{dx} = \frac{1}{x}$$



Field of Slopes



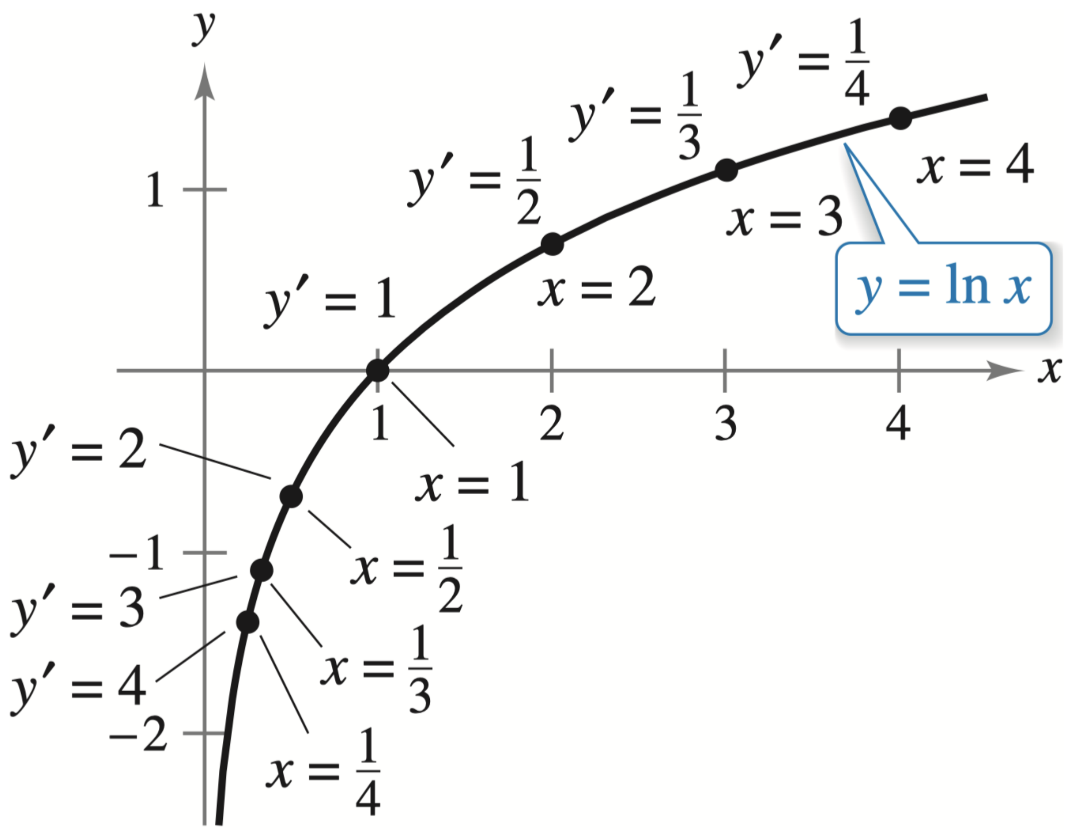
Each small line segment has a slope of $\frac{1}{x}$.

Properties of the Natural Logarithmic Function

1. Domain:
   Range:

2. The function is continuous, increasing, and one-to-one.

3. The graph is concave downward.



Logarithmic Properties

If a and b are positive numbers and n is rational, then the following properties are true.

1. $\ln(1) = 0$
2. $\ln(ab) = \ln(a) + \ln(b)$
3. $\ln(a^n) = n\ln(a)$
4. $\ln(\frac{a}{b}) = \ln(a) - \ln(b)$

Examples:

a. $\ln(\frac{10}{9})$

b. $\ln(\sqrt{3x+2})$

c. $\ln(\frac{6x}{5})$

d. $\ln\left(\frac{(x^2+3)^2}{x\sqrt[3]{x^2+1}}\right)$

Practice

A Use the laws of logarithms to expand each expression.

1. $\ln \dfrac{(x^2+5)^4 \sin(x)}{x^3+1}$
2. $\ln \dfrac{r^2}{3\sqrt{s}}$
3. $\ln \sqrt{a(b^2+c^2)}$
4. $\ln(uv)^{10}$
5. $\ln \dfrac{3x^2}{(x+1)^5}$

B Express the expression as a single logarithm.

6. $\ln 3 + \dfrac{1}{3}\ln 8$
7. $\ln(1+x^2) + \dfrac{1}{2}\ln(x) - \ln(\sin(x))$
8. $\ln(a+b) + \ln(a-b) - 2\ln c$

The number e

The letter e denotes the positive real number such that

$$\ln(e) = \int_1^e \frac{1}{t}\,dt$$