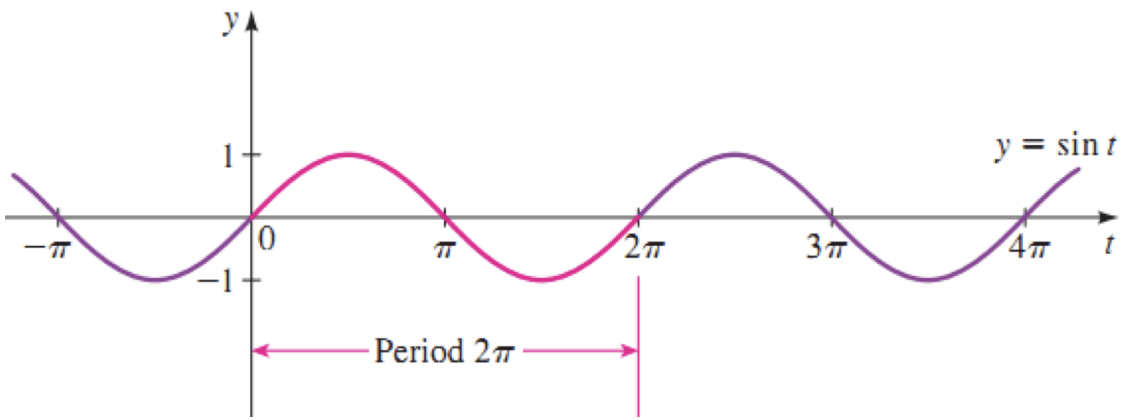
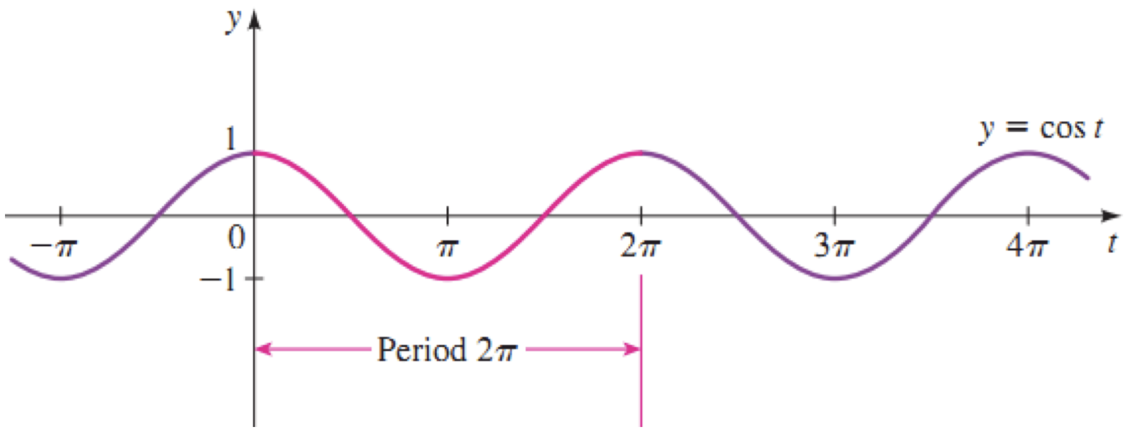
How are the ideas of inverse functions applied to find the derivatives and integrals of inverse trigonometric functions?

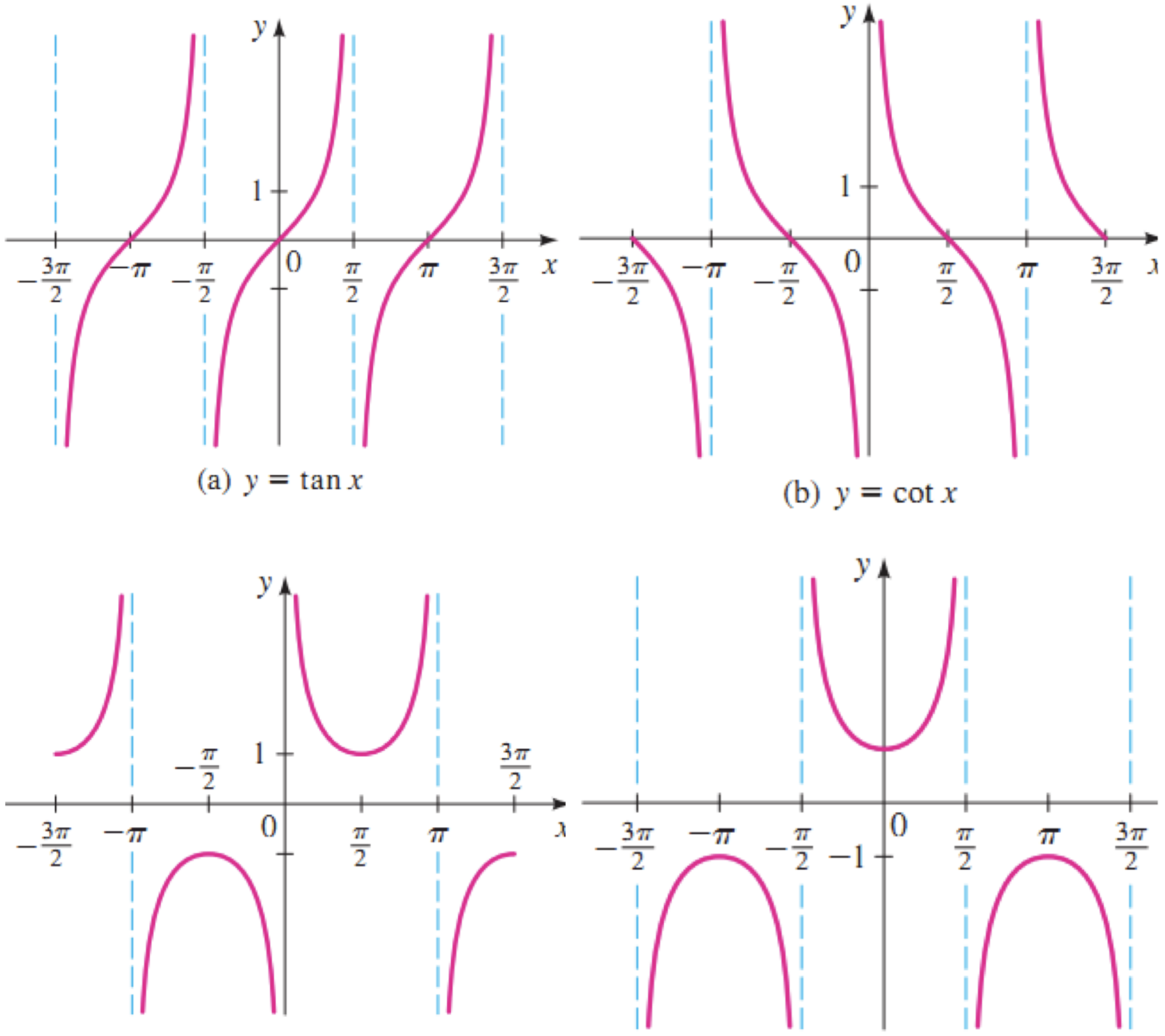
Quick Check

Determine if the following functions have inverses.





Graphs of other trignometric functions



(a) $y = \tan x$

(b) $y = \cot x$

(c) $y = \csc x$

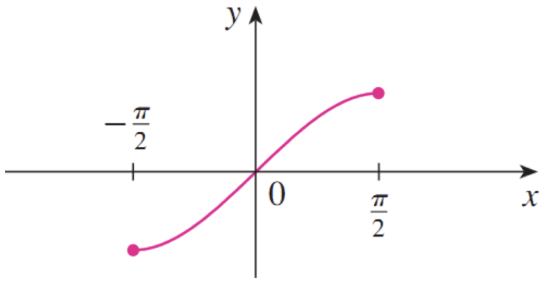
(d) $y = \sec x$

Definition of Inverse Sine Function

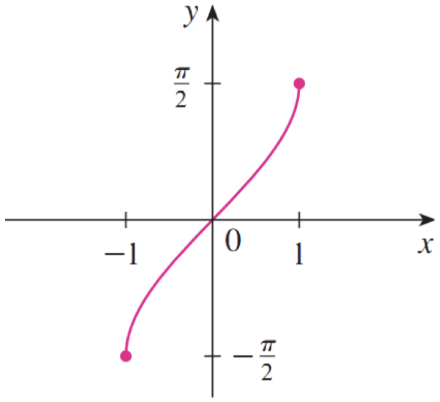
The *inverse sine function* is the function $\sin^{-1}$ with the domain $[-1, 1]$ and the range $[\frac{-\pi}{2}, \frac{\pi}{2}]$ defined by

$$\sin^{-1} x = y \iff \sin(y) = x$$

The inverse sine function is also denoted by $arcsin$.



$y = \sin x, -\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$



$y = \sin^{-1} x = \arcsin x$

🤔 Build the derivative

Derivative of inverse sine function

$$\frac{d}{dx}\left[\sin^{-1} u\right] = \frac{1}{\sqrt{1-u^2}} \cdot u'$$

Example

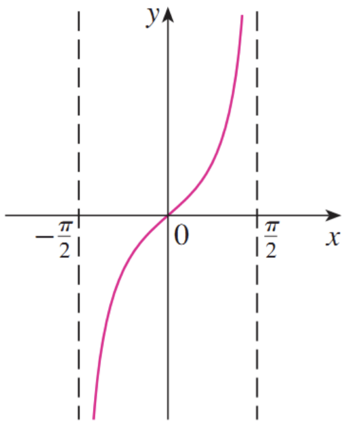
$$\frac{d}{dx}\left[\arcsin(2x)\right] =$$

Definition of Inverse Tangent Function

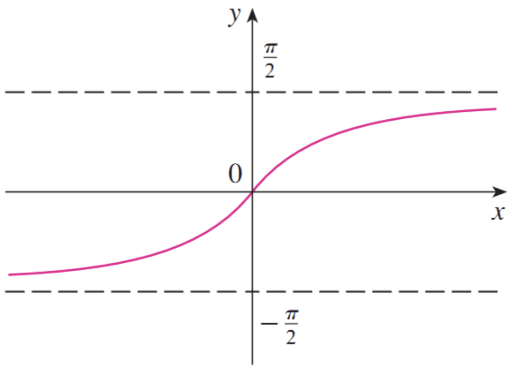
The *inverse tangent function* is the function $\tan^{-1}$ with the domain $[-\infty, \infty]$ and the range $[\frac{-\pi}{2}, \frac{\pi}{2}]$ defined by

$$\tan^{-1} x = y \iff \tan(y) = x$$

The inverse tangent function is also denoted by $arctan$.



$y = \tan x, -\frac{\pi}{2} < x < \frac{\pi}{2}$



$y = \tan^{-1} x = \arctan x$

🤔 Build the derivative

Derivative of inverse tangent function

$$\frac{d}{dx}\left[\tan^{-1} u\right] = \frac{1}{1+u^2} \cdot u'$$

Example

$$\frac{d}{dx}\left[\arctan(x^2)\right] =$$

Practice

Find the derivative of each function.

1. $f(t) = \arcsin(t^2)$
2. $f(x) = \arctan\left(\sqrt{x}\right)$
3. $f(t) = \cos\left(\arccos t\right)$
4. $f(x) = x^2 \arctan x$
5. $y = x\arctan(2x) - \dfrac{1}{4}\ln(1 + 4x^2)$

Integrals involving Inverse Trigonometric Functions

Let u be a differentiable function of x, and let $a > 0$.

1 $$\int \frac{1}{\sqrt{a^2 - u^2}}\,du = \arcsin\left(\frac{u}{a}\right) + C$$

2 $$\int \frac{1}{a^2 + u^2}\,du = \frac{1}{a}\arctan\left(\frac{u}{a}\right) + C$$

Examples

1. $\displaystyle\int \frac{1}{\sqrt{4 - x^2}}\,dx$

2. $\displaystyle\int \frac{1}{\sqrt{2 - 9x^2}}\,dx$

Practice

Find the integral.

1. $$\int \frac{7}{16 + x^2}\, dx$$

2. $$\int \frac{3}{\sqrt{1 - 4x^2}}\, dx$$

3. $$\int \frac{t}{t^4 + 16}\, dt$$

4. $$\int \frac{t}{\sqrt{1 - t^4}}\, dt$$

Comparing Integration Problems

Find as many of the following integrals as you can using the formulas and techniques you have studied so far in the text.

a. $\displaystyle\int \frac{dx}{x\sqrt{x^2-1}}$ **b.** $\displaystyle\int \frac{x\,dx}{\sqrt{x^2-1}}$ **c.** $\displaystyle\int \frac{dx}{\sqrt{x^2-1}}$

Solution

a. You *can* find this integral (it fits the Arcsecant Rule).

$$\int \frac{dx}{x\sqrt{x^2-1}} = \operatorname{arcsec}|x| + C$$

b. You *can* find this integral (it fits the Power Rule).

$$\begin{aligned}\int \frac{x\,dx}{\sqrt{x^2-1}} &= \frac{1}{2}\int (x^2-1)^{-1/2}(2x)\,dx \\ &= \frac{1}{2}\left[\frac{(x^2-1)^{1/2}}{1/2}\right] + C \\ &= \sqrt{x^2-1} + C\end{aligned}$$

c. You *cannot* find this integral using the techniques you have studied so far. (You should scan the list of basic integration rules to verify this conclusion.)

Comparing Integration Problems

Find as many of the following integrals as you can using the formulas and techniques you have studied so far in the text.

a. $\int \frac{dx}{x \ln x}$ **b.** $\int \frac{\ln x\, dx}{x}$ **c.** $\int \ln x\, dx$

Solution

a. You *can* find this integral (it fits the Log Rule).

$$\int \frac{dx}{x \ln x} = \int \frac{1/x}{\ln x}\, dx$$
$$= \ln|\ln x| + C$$

b. You *can* find this integral (it fits the Power Rule).

$$\int \frac{\ln x\, dx}{x} = \int \left(\frac{1}{x}\right)(\ln x)^1\, dx$$
$$= \frac{(\ln x)^2}{2} + C$$

c. You *cannot* find this integral using the techniques you have studied so far. ■