How can we use partial fractions to aid integration?

Quick Check

Combine the following functions

$$\frac{1}{x-2}+\frac{-1}{x+3}$$

JOHN BERNOULLI (1667–1748)

The method of partial fractions was introduced by John Bernoulli, a Swiss mathematician who was instrumental in the early development of calculus. John Bernoulli was a professor at the University of Basel and taught many outstanding students, the most famous of whom was Leonhard Euler.

How to reverse the process? 🤔

$$\frac{5}{x^2+x-6} = \frac{?}{(x-2)} + \frac{?}{x+3}$$

Notice, integrating $\dfrac{5}{x^2+x-6}$ is hard. However, integrating $\dfrac{1}{x-2} + \dfrac{-1}{x+3}$ is easy.

The method of partial fractions shows how to reverse the process. ❗ Although, this method is just good for rational functions whose denominators factor nicely.

🧰 Keep your Integration Toolbox handy

Just because it's an integral of a rational function, don't start thinking partial fractions.

Think about how you would tackle the following integral.

$$\int \frac{x^2 + x + 2}{x^2 + 1}\, dx$$

Method of Partial Fractions

$$\int \frac{1}{x^2 - 5x + 6}\,dx$$

Start by factoring the denominator.

$$\frac{1}{x^2 - 5x + 6} = \frac{1}{(x-3)(x-2)} = \frac{A}{x-3} + \frac{B}{x-2}$$

Now solve for A and B. Then, ofcourse return to integrating.

Decomposition

Helper Rules for decomposing $N(x)/D(x)$ into partial fractions

1. Divide when improper: $\dfrac{N(x)}{D(x)} = \text{a polynomial} + \dfrac{N_1(x)}{Dx}$

2. Factor Denominator completely

3. Linear Factors: $\dfrac{A}{(px+q)} + \dfrac{B}{(px+q)^2} + \cdots + \dfrac{A}{(px+q)^n}$

4. Quadratic Factors: $\dfrac{Ax+B}{ax^2+bx+c} + \dfrac{Ax+B}{(ax^2+bx+c)^2} + \cdots + \dfrac{Ax+B}{(ax^2+bx+c)^n}$

Examples

Distinct Linear Factors

1 $$\int \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x}\, dx$$

Distinct Linear and Quadratic Factors

3 $$\int \frac{2x^3 - 4x - 8}{(x^2 - x)(x^2 + 4)}\, dx$$

Repeated Quadratic Factors

4 $$\int \frac{8x^3 + 13x}{(x^2 + 2)^2}\, dx$$

You Try

2 $$\int \frac{5x^2 + 20x + 6}{x^3 + 2x^2 + x}\, dx$$

Practice

1. $$\int \frac{5x^2 - 12x - 12}{x^3 - 4x}\,dx$$

2. $$\int \frac{x^2 + 3x - 4}{x^3 - 4x^2 + 4x}\,dx$$

3. $$\int \frac{6x}{(x-2)(x^2+2x+4)}\,dx$$

4. $$\int \frac{x^2 - x + 9}{(x^2+9)^2}\,dx$$

Logic for repeating factors.

Authored as a response on StackExchange.

Consider the simplest of cases.

$$\frac{\xi}{x^3}$$

Assuming ξ is some polynomial, one could carry on long division to determine the quotient and the remainder. Thus one could get

$$\frac{\xi}{x^3} = q(x) + \frac{r(x)}{x^3}$$

Now what can we say for certain about $r(x)$? We can say that it is the remainder thus of smaller degree than x^3. The most general possible polynomial of degree 2 or smaller is $A + Bx + Cx^2$. Then

$$\frac{r(x)}{x^3} = \frac{A + Bx + Cx^2}{x^3}$$

Further more, we can split the right side

$$\frac{r(x)}{x^3} = \frac{A}{x^3} + \frac{Bx}{x^3} + \frac{Cx^2}{x^3}$$

which becomes

$$\frac{r(x)}{x^3} = \frac{A}{x^3} + \frac{B}{x^2} + \frac{C}{x}$$

Thus the above would be a sensible what to try to split any proper fraction $\frac{r(x)}{x^3}$ Moreover, the similar idea would hold for $\frac{r(x)}{(x+a)^3}$, meaning a sensible way to split it would be

$$\frac{r(x)}{(x+a)^3} = \frac{A}{(x+a)^3} + \frac{B}{(x+a)^2} + \frac{C}{(x+a)}$$

Imagine you tried to do partial fractions on $\frac{1+x}{x^2}$. You could try $\frac{1+x}{x^2} = \frac{A}{x} + \frac{B}{x}$, but that's just $\frac{A+B}{x}$ on the right so it won't work. However, $\frac{1+x}{x^2} = \frac{1}{x^2} + \frac{x}{x^2}$, which simplifies into $\frac{1}{x^2} + \frac{1}{x}$, which is the best we can do. – Akiva Weinberger Jan 7 '16 at 17:44