What are improper integrals and how do we evaluate them?

Quick Check

Recall $\int_a^b f(x)dx = F(b) - F(a)$

Describe why the statement is incorrect.

$$\int_{-2}^{1} \frac{2}{x^3}dx = \left[-\frac{1}{x^2}\right]_{-2}^{1} = -\frac{3}{4}$$

Fundamental Theorem of Calculus

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Requires the interval [a,b] to be finite

Function needs to be continuous

How do we evaluate integrals that do not satisfy these requirements?

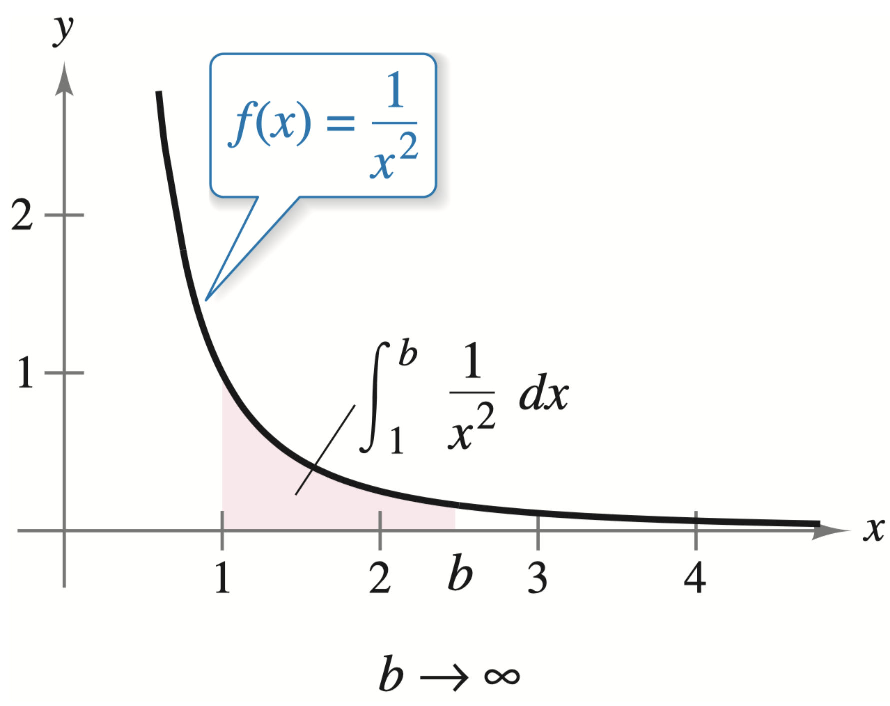
Improper Integrals

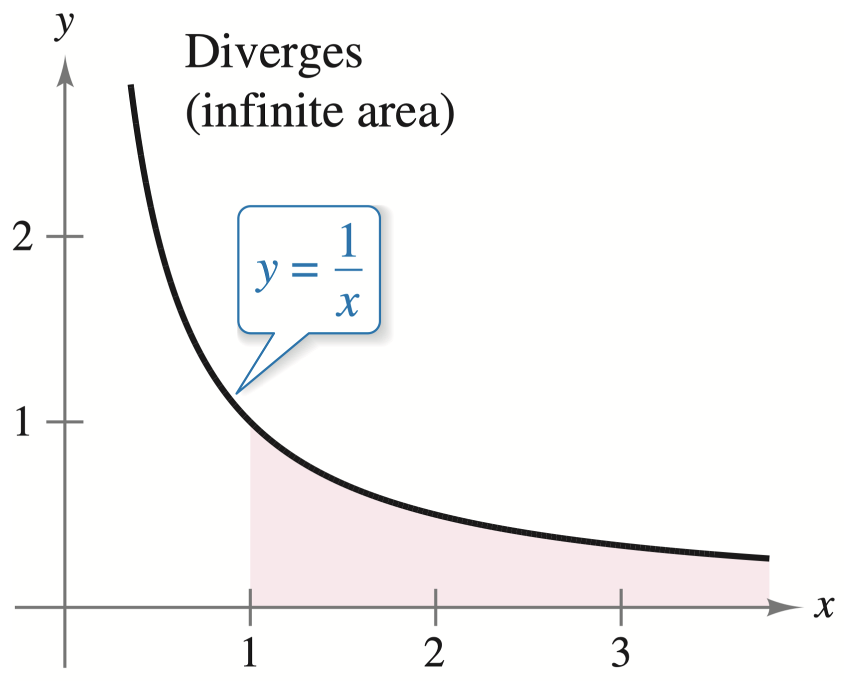
Integrals whose upper or lower limits are infinite

We are integrating a function with an infinite discontinuity

Example:

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty}\left(\int_1^b \frac{1}{x^2}\,dx\right) = \lim_{b\to\infty}\left(1 - \frac{1}{b}\right) = 1$$



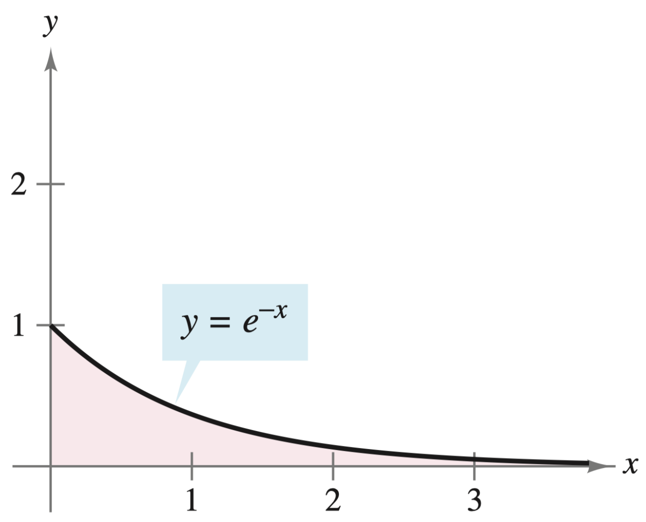


An Improper Integral that Diverges

Evaluate $\int_1^{\infty} \frac{1}{x}\,dx$

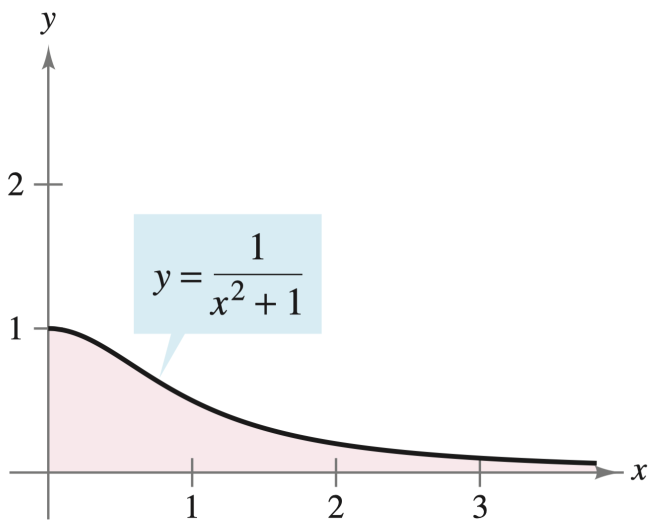
Improper Integrals that Converge

1 $\int_0^{\infty} e^{-x}\,dx$



The area of the unbounded region is 1.

2 $\int_0^{\infty} \frac{1}{x^2+1}\,dx$



The area of the unbounded region is $\pi/2$.

Using L'hopital's Rule with an Improper Integral

1 $\int_0^{\infty} (1-x)e^{-x}\,dx$

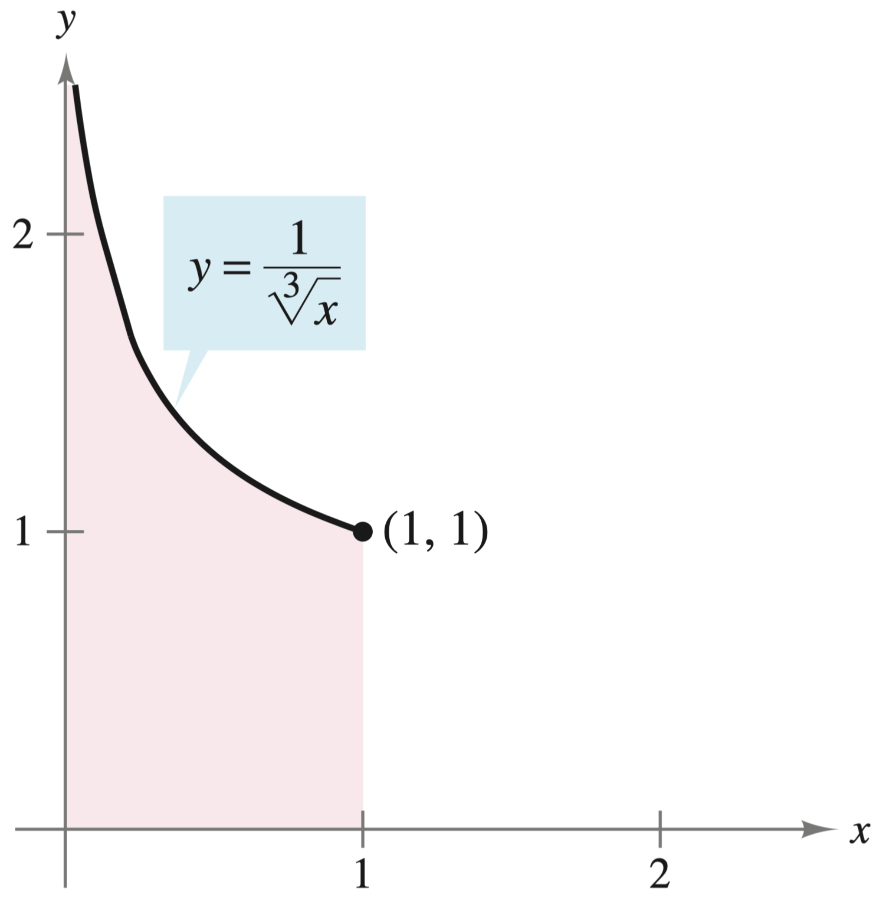
2 $\int_{-\infty}^{\infty} \frac{e^x}{1+e^{2x}}\,dx$

Improper integrals with an infinite discontinuity

$$\int_0^1 \frac{1}{\sqrt[3]{x}}\,dx = \lim_{b\to 0^+}\left[\frac{x^{2/3}}{2/3}\right]$$

$$= \lim_{b\to 0^+} \frac{3}{2}(1 - b^{2/3})$$

$$= \frac{3}{2}$$



Infinite discontinuity at $x = 0$

Interior Discontinuity

1 $\int_{-1}^{2} \frac{1}{x^3}\, dx$

Draw a Graph!

2 Gabriel's Horn - finite volume and infinite surface area.

Find the volume of the solid formed by revolving (about the x-axis) the unbounded region lying between the graph of $f(x) = \frac{1}{x}$ and the x-axis for $x \geq 1$.