How do you determine if a sequence converges or diverges?

Quick Check

Describe a pattern for each of the following sequences. Then use your description to write a formula for the n^{th} term of each sequence.

1. $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$

2. $1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \frac{1}{120}, \ldots$

3. $10, \frac{10}{3}, \frac{10}{6}, \frac{10}{10}, \frac{10}{15}, \ldots$

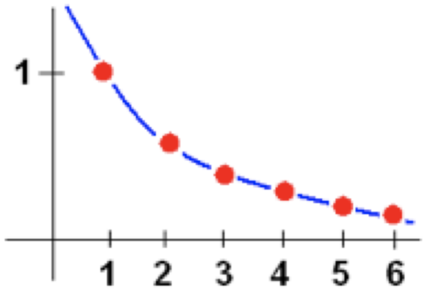
4. $\frac{1}{4}, \frac{4}{9}, \frac{9}{16}, \frac{16}{25}, \frac{25}{36}, \ldots$

Sequence

A sequence is defined as a function whose domain is the set of positive integers.

$$f(x) \to a(x) \to a(n) \to a_n$$

$$f(x) = \frac{1}{x} \to a_n = \frac{1}{n}$$



$f(x)$ **is continuous**

a_n **is discrete**

Other notation: $\{a_n\}_1^{\infty}$ or simply just $\{a_n\}$

expanded form: $a_1, a_2, a_3, \ldots a_n, \ldots$

nth term

In our example: $\left\{\frac{1}{n}\right\} = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots$

Understanding the formula for a sequence

List the first four terms of each sequence.

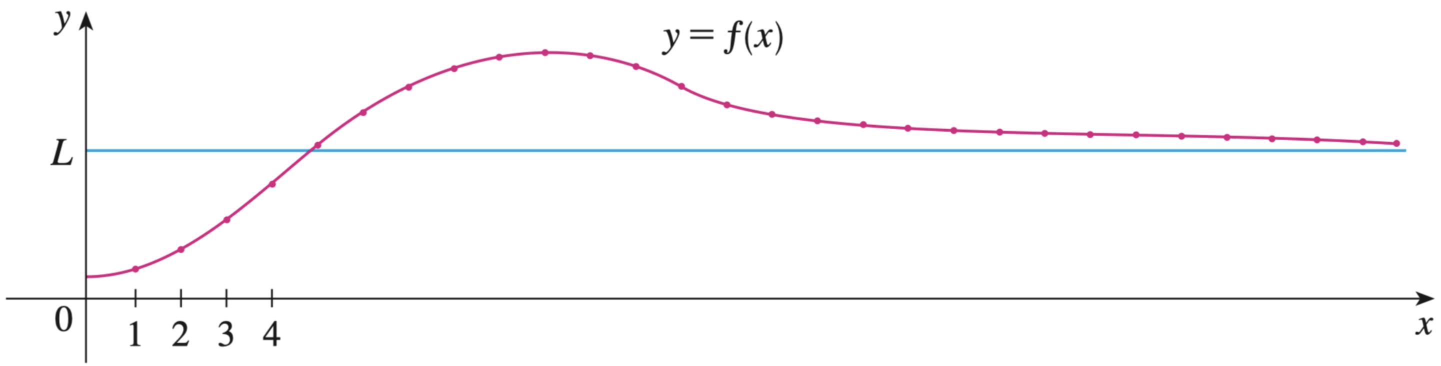
1 $a_n = \{3 + (-1)^n\}$

2 $b_n = \left\{\dfrac{n}{1 - 2n}\right\}$

3 $c_n = \left\{\dfrac{n^2}{2^n - 1}\right\}$

4 Recursively defined sequence d_n, where $d_1 = 25$ and $d_{n+1} = d_n - 5$

Limit of a Sequence



Let L be a real number. Let f be a function of a real variable such that

$$\lim_{x \to \infty} f(x) = L$$

If a_n is a sequence such that $f(n) = a_n$ for every positive integer n, then

$$\lim_{n \to \infty} a_n = L$$

Properties of Limits of Sequences

Let $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = K$.

1. $\lim_{n\to\infty} (a_n \pm b_n) = L \pm K$

2. $\lim_{n\to\infty} ca_n = cL,$ c is any real number

3. $\lim_{n\to\infty} (a_n b_n) = LK$

4. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{K},$ $b_n \neq 0$ and $K \neq 0$

Find the limit of the sequence.

$$a_n = \left(1 + \frac{1}{n}\right)^n$$

Convergent or Divergent

1 $a_n = \{3 + (-1)^n\}$

2 $b_n = \left\{ \dfrac{n}{1 - 2n} \right\}$

3 $c_n = \left\{ \dfrac{n^2}{2^n - 1} \right\}$

4 $b_n = \left\{ (-1)^n \dfrac{1}{n!} \right\}$

Try Squeeze Theorem

Absolute Value Theorem

For the sequence $\{a_n\}$, if

$$\lim_{n\to\infty} |a_n| = 0 \qquad \text{then} \qquad \lim_{n\to\infty} a_n = 0.$$

Results follow by the squeeze theorem.

$$-|a_n| \leq a_n \leq |a_n|$$

Find the n^{th} term of a Sequence

Find a sequence a_n whose first five terms are

$$\frac{2}{1}, \frac{4}{3}, \frac{8}{5}, \frac{16}{7}, \frac{32}{9}, \ldots$$

and then determine if the sequence converges or diverges.

It is not possible to determine convergence from the first few terms...we really need a general formula.

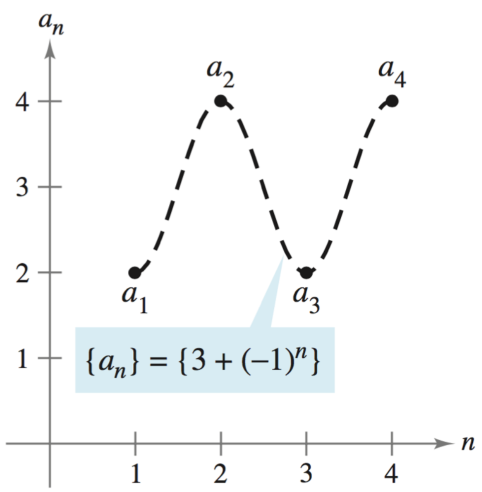
Monotonic Sequences and Bounded Sequences

A sequence a_n is monotonic if its terms are nondecresing.

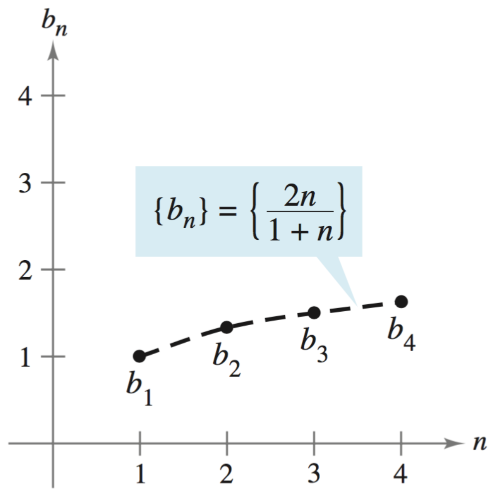
$$a_1 \leq a_2 \leq a_3 \leq \ldots \leq a_n \leq \ldots$$

or if its terms are nonincreasing

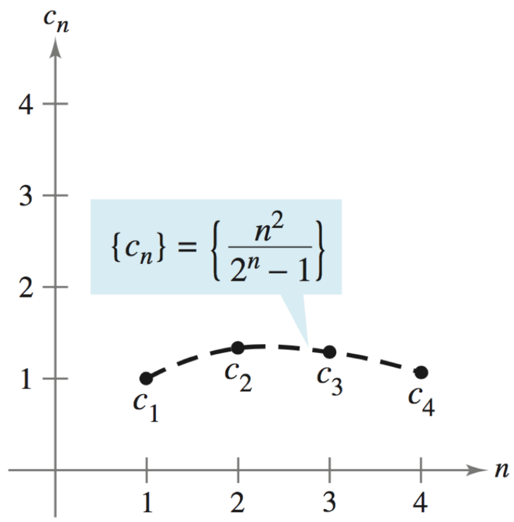
$$a_1 \geq a_2 \geq a_3 \geq \ldots \geq a_n \geq \ldots$$



(a) Not monotonic



(b) Monotonic



(c) Not monotonic

Determine if a sequence is monotonic

1. $a_n = 3 + (-1)^n$

2. $b_n = \left\{ \dfrac{2n}{1+n} \right\}$

3. $c_n = \left\{ \dfrac{n^2}{2^n - 1} \right\}$

Definition of a Bounded Sequence

1. A sequence $\{a_n\}$ is **bounded above** if there is a real number M such that $a_n \leq M$ for all n. The number M is called an **upper bound** of the sequence.
2. A sequence $\{a_n\}$ is **bounded below** if there is a real number N such that $N \leq a_n$ for all n. The number N is called a **lower bound** of the sequence.
3. A sequence $\{a_n\}$ is **bounded** if it is bounded above and bounded below.

Bounded Monotonic Sequences

Theorem: If a sequence is bounded and monotonic, then it converges. 🤔

Examples: Which sequence is convergent? What does this mean?

1️⃣ $a_n = \{1/n\}$

2️⃣ $b_n = \{n^2/(n+1)\}$

3️⃣ $c_n = \{(-1)^n\}$

Practice

Find the limit (if possible) of the sequence.

1. $a_n = \dfrac{5n^2}{n^2 + 2}$

2. $a_n = \dfrac{2n}{\sqrt{n^2 + 1}}$

3. $b_n = \sin \dfrac{1}{n}$

4. $c_n = \dfrac{1 \cdot 3 \cdot 5 \cdots (2n - 1)}{(2n)^n}$