What does it mean to say that a series converges?

Quick Check - True / False. Explain.

1. If $\{a_n\}$ converges to 3 and $\{b_n\}$ converges to 2, then $\{a_n + b_n\}$ converges to 5.

2. If $\{a_n\}$ converges, then $\{\frac{a_n}{n}\}$ converges to 0.

3. If $\{a_n\}$ converges, then $\lim_{n\to\infty}(a_n - a_{n-1}) = 0$.

Series

If $\{a_n\}$ is an infinite sequence, then

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots + a_n + \ldots$$

To find the sum of the infinite series, consider the sequence of partial sums

$S_1 = a_1$

$S_2 = a_1 + a_2$

$S_3 = a_1 + a_2 + a_3$ $\qquad n^{th}$ partial sum

$\vdots$

$S_n = a_1 + a_2 + a_3 + \ldots + a_n$

If this sequence of partial sums converges, the series is said to converge.

Convergent and Divergent Series

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$$

$$S_1 =$$

$$S_2 =$$

$$S_3 =$$

$$\vdots$$

$$S_n =$$

Does the sequence of partial sums converge? What is the sum of the infinite series?

Partial Sums and Convergence

$\sum_{n=1}^{\infty} 1 = 1 + 1 + 1 + 1 + 1 + 1 + \ldots$ converges or diverges? What is your reasoning?

Telescoping Series

The n^{th} partial sum of the series

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)=+\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\dots$$

is given by

$$s_n = 1-\frac{1}{n+1}$$

Does this series converge? If so, what is the sum?

Telescoping series

$(b_1 - b_2) + (b_2 - b_3) + (b_3 - b_4) + (b_4 - b_5) + \dots$ notice telescoping terms

Observe n^{th} partial sum is $S_n = b_1 - b_{n+1}$. Telescoping series will converge if and only if b_n approaches a finite number as $n \to \infty$. If the series converges, its sum is

$$S = b_1 - \lim_{n\to\infty} b_{n+1}$$

Find the sum of the series.

$$\sum_{n=1}^{\infty} \frac{2}{4n^2 - 1}$$

Geometric Series

$$\sum_{n=1}^{\infty} ar^n = a + ar + ar^2 + \cdots + ar^n + \dots \quad a \neq 0$$

1. A geometric series with ratio r diverges if $|r| \geq 1$.

2. If $0 < |r| < 1$, then the series converges to the sum

$$\sum_{n=1}^{\infty} ar^n = \frac{a}{1-r}$$

Convergent and Divergent Geometric Series

1 $\displaystyle\sum_{n=0}^{\infty} \frac{3}{2^n}$

Common Ratio $=$

$a =$

$S = \dfrac{a}{1-r} =$

2 $\displaystyle\sum_{n=0}^{\infty} \left(\frac{3}{2}\right)^n$

Common Ratio $=$

$a =$

$S = \dfrac{a}{1-r} =$

Repeating Decimal $\rightarrow$ geometric series

Use a geometric series to write $0.0\overline{8}$ as the ratio of two integers.

Properties of Infinite Series

If $\sum a_n = A, \quad \sum b_n = B$, and c is a real number, then the following series converge to the indicated sums.

1. $$\sum_{n=1}^{\infty} c \cdot a_n = cA$$

2. $$\sum_{n=1}^{\infty} a_n + b_n = A + B$$

3. $$\sum_{n=1}^{\infty} a_n - b_n = A - B$$

Thinking about convergence of a series 🤔

If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n \to \infty} a_n = 0$.

⚠️ If a sequence a_n converges to 0, then the series may either converge or diverge.

$n^{th}-$ Term test for Divergence

If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Examples:

1. $\sum_{n=0}^{\infty} 2^n$
2. $\sum_{n=1}^{\infty} \frac{n!}{2n!+1}$
3. $\sum_{n=1}^{\infty} \frac{1}{n}$

Practice

Determine the convergence or divergence of the series using any method.

1. $$\sum_{n=1}^{\infty} \frac{n+10}{10n+1}$$

2. $$\sum_{n=1}^{\infty} \frac{1}{n \cdot (n+3)}$$

3. $$\sum_{n=0}^{\infty} \frac{4}{2^n}$$