What does it mean to say that a series is 'conditionally convergent' or 'absolutely convergent'?

Quick Check

Write out a few terms of the following series. Are the terms all positive, negative, or alternating? Explain.

$$\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$$

Absolute Convergence

If the series $\sum |a_n|$ converges, then the series $\sum a_n$ also converges.

⚠️ The converse is not true.

Example

$$\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$$

Absolute and Conditional Convergence

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

The alternating harmonic series converges by the alternating series test but $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges. Therefore, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is conditionally convergent.

1. $\sum a_n$ is absolutely convergent if $\sum |a_n|$ converges.
2. $\sum a_n$ is conditionally convergent if $\sum a_n$ converges but $\sum |a_n|$ diverges.

Practice

Determine whether each of the series is convergent or divergent. Classify any convergent series as absolutely or conditionally convergent.

1. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n n!}{2^n}$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n(n+1)/2}}{3^n}$

Practice

Determine whether the series converges conditionally or absolutely, or diverges.

1. $$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(n+1)^2}$$

2. $$\sum_{n=2}^{\infty} \frac{\cos n\pi}{n+1}$$

Ratio Test

Let $\sum a_n$ be a series with nonzero terms.

1. $\sum a_n$ converges absolutely if $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1.$

2. $\sum a_n$ diverges if $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1$ or $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$

3. The Ratio Test is inclusive if $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$

Example and Practice

Determine the convergence and divergence.

1 $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{n!}$

2 $\displaystyle\sum_{n=0}^{\infty} \frac{n^2 2^{n+1}}{3^n}$

3 $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!}$

4 $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{n+1}$

Root Test

Let $\sum a_n$ be a series.

1. $\sum a_n$ converges absolutely if $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$

2. $\sum a_n$ diverges if $\lim_{n\to\infty} \sqrt[n]{|a_n|} > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$

3. The Root Test is inconclusive if $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$

Determine the convergence or divergence of $\sum_{n=1}^{\infty} \frac{e^{2n}}{n^n}$.

Guidelines for testing a series for convergence or divergence

1. Does the n^{th} term approach 0? If not, the series diverges.

2. Is the series one of the special types - geometric, $p-$series, telescoping, or alternating?

3. Can the Integral Test, the Root Test, or the Ratio Test be applied?

4. Can the series be compared favorably to one of the special types?

1 $\sum_{n=1}^{\infty} \frac{n+1}{3n+1}$

2 $\sum_{n=1}^{\infty} \left(\frac{\pi}{6}\right)^n$

3 $\sum_{n=1}^{\infty} ne^{-n^2}$

4 $\sum_{n=1}^{\infty} (-1)^n \frac{3}{4n+1}$

5 $\sum_{n=1}^{\infty} \frac{n!}{10^n}$

6 $\sum_{n=1}^{\infty} \left(\frac{n+1}{2n+1}\right)^n$