How can polynomial functions be used as approximations for other elementary functions?

Quick Check

Let f be a differentiable function such that $f(3)=2$ and $f'(3)=5$. If the tangent line to the graph of f at $x=3$ is used to find an approximation to a zero of f, that approximation is

$a)$ 0.4 $b)$ 0.5 $c)$ 2.6 $d)$ 3.4 $e)$ 5.5

Polynomial Approximation

Let's say we want to approximate $f(x) = \sin x$ by a polynomial. draw sketches

- We set a requirement that this polynomial has to go through one point $(0, 0)$ on $f(x) = \sin x$. $\sin(0) = P(0)$ but this polynomial doesn't provide a good estimate for f anywhere else.

- So we set a second requirement for this polynomial.

1. It has to go though the point $(0, 0)$.
2. It has to have the same derivative (go with the flow of f) at $x = 0$.

This is the tangent line approximation. The polynomial approximates the sine function nicely in the vicinity of "center" $(0, 0)$ but not much further.

Polynomial Approximation (continued)

- So we set a 3^{rd} requirement for the polynomial.
  1. It has to go though the point $(0, 0)$.
  2. It has to have the same first derivative $\left(\text{go with the flow of } f\right)$ at $x = 0$.
  3. It has to have the same 2^{nd} derivative $\left(\text{bend the same as } f\right)$ at $x = 0$.

This polynomial approximation is even better in the neighborhood of the center $(0, 0)$ where it is based.

$\rightarrow$ In the same manner, more requirements will make the polynomial approximation better and better. (View on Geogebra)

Taylor Polynomial - problem of coefficients

Let's say we have a function $f(x)$, we want to approximate with a $P_n(x)$, a polynomial, with center at $x = 0$.

$$f(x) \approx P_n(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots + a_n x^n$$
$$P_n(0) = a_0$$

$$a_0 = P_n(0)$$

$$P_n'(x) = a_1 + 2 \cdot a_2 x + 3a_3 \cdot x^2 + 4a_4 \cdot x^3 + \dots + n a_n x^{n-1}$$
$$P_n'(0) = a_1$$

$$a_1 = \frac{P_n'(0)}{1!}$$

$$P_n''(x) = 2 \cdot a_2 + 2 \cdot 3a_3 \cdot x + 3 \cdot 4a_4 \cdot x^2 + \dots + n \cdot (n-1) a_n x^{n-2}$$
$$P_n''(0) = 2a_2$$

$$a_2 = \frac{P_n''(0)}{2!}$$

$$P_n'''(x) = 2 \cdot 3a_3 \cdot + 2 \cdot 3 \cdot 4a_4 \cdot x + \dots + n \cdot (n-1) \cdot (n-2) a_n x^{n-3}$$
$$P_n'''(0) = 2 \cdot 3 \cdot a_3$$

$$a_3 = \frac{P_n'''(0)}{3!}$$

$$\vdots$$

$$P_n^{(n)}(x) = n! a_n$$
$$P_n^{(n)}(0) = n! a_n$$

$$a_n = \frac{P_n^{(n)}(0)}{n!}$$

Taylor Polynomial and Maclaurin Polynomial

If f has n derivatives at c, then the polynomial

$$P_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^n(c)}{n!}(x-c)^n$$

is called the n^{th} Taylor polynomial for f at c.

If $c = 0$, then

$$P_n(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \cdots + \frac{f^n(0)}{n!}(x-0)^n$$

is also called the n^{th} Maclaurin polynomial for f.

Finding Taylor Polynomials

1. Find T_6 and T_n for $f(x) = e^x$ centered at $c = 1$.

2. Find T_3 and T_n for $f(x) = e^x$ centered at $c = 0$.

3. Find the n^{th} Maclaurin polynomial for $\cos x$. center??

4. Find the Taylor polynomial P_4 for $f(x) = \ln x$ centered at $c = 1$.

5. Find the T_4 Taylor Polynomial for $f(x) = \dfrac{1}{x^2}$ centered at $c = 2$.

How accurate is our polynomial approximation of a function? 🤔

$$f(x) = P_n(x) + R_n(x)$$

Exact value — Approximate Value — Remainder

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^n(c)}{n!}(x-c)^n + R_n(x)$$

The remainder estimate can be found as follows

$|R_n(x)| \leq \dfrac{max|f^{n+1}(z)|}{(n+1)!}(x-c)^{n+1}$ for any z between x and c.

Taylor Remainder Estimates

1 Find the 3^{rd} degree Maclaurin Polynomial for $\sin x$.

2 Using (1) approximate $\sin(.1)$ by $P_3(.1)$ and determine the accuracy of the result.

3 Find 5^{th} degree Maclaurin Polynomial for e^x.

4 Using (3) estimate $|e^1 - P_5(1)|$.

5 What degree Maclaurin polynomial can approximate $e^{.3}$ to $.001$.