What is a power series?

Quick Check

Write down each Maclaurin Polynomial Approximation

$e^x \approx$ 1st degree polynomial

$e^x \approx$ 2st degree polynomial

$e^x \approx$ 3st degree polynomial

$e^x \approx$ 4st degree polynomial

e^x can be represented exactly by a *power series*

e^x = 🤔

Definition of Power Series

If x is a variable, then an infinite series of the form

$$\sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c)^1 + a_2(x-c)^2 + \cdots + a_n(x-c)^n + \dots$$

is called a power series centered at c, where c is a constant.

🤔 What would be the expression for a power series cented at 0 ($c = 0$)?

Examples of Power Series

1 $$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots$$

2 $$\sum_{n=0}^{\infty} (-1)^n (x+1)^n = 1 - (x+1) + (x+1)^2 + (x+1)^3 + \dots$$

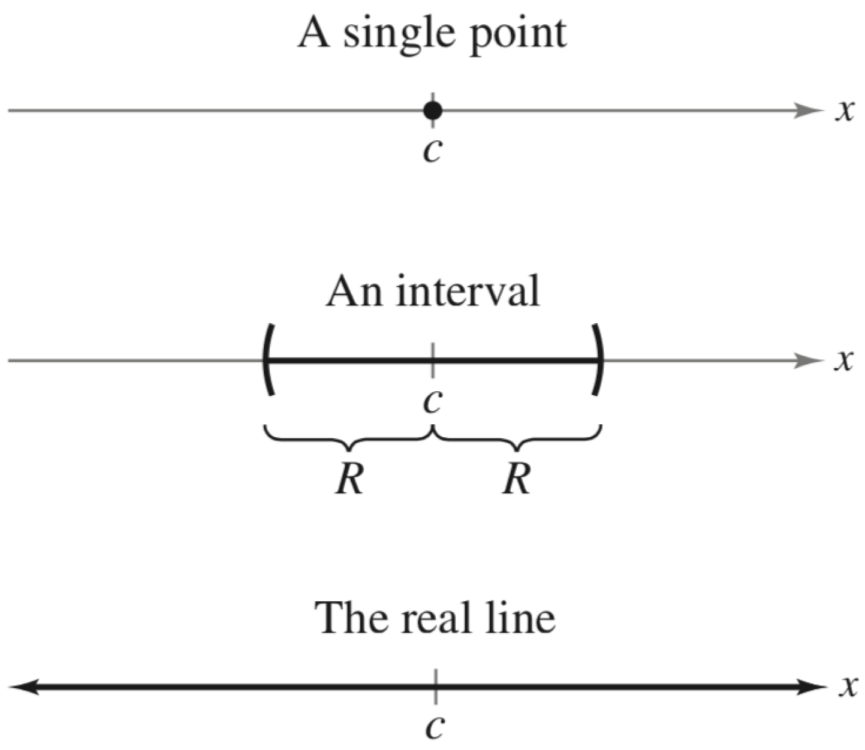
3 $$\sum_{n=1}^{\infty} \frac{1}{n}(x-1)^n = (x-1) + \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 + \dots$$

Domain of a Power Series

A power series in x can be viewed as a function of x: → $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$

The domain of the power series has three basic forms:

1. A point (center)
2. An interval (🔍 consider endpoints)
3. Entire real line



Radius of Convergence

Find the radius and the interval of convergence.

1. $\sum_{n=0}^{\infty} n!x^n$

2. $\sum_{n=0}^{\infty} 3(x-2)^n$

3. $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$

4. $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

5. $\sum_{n=0}^{\infty} \frac{(-1)^n (x+1)^n}{2^n}$

Properties of functions defined by power series.

If the function given by

$$f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$$
$$= a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots$$

has a radius of convergence of $R > 0$, then, on the interval $(c - R, c + R)$, f is differentiable (and therefore continuous). Moreover, the derivative and antiderivative of f are as follows.

1. $$f'(x) = \sum_{n=1}^{\infty} na_n(x-c)^{n-1}$$
$$= a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \cdots$$

2. $$\int f(x)\,dx = C + \sum_{n=0}^{\infty} a_n \frac{(x-c)^{n+1}}{n+1}$$
$$= C + a_0(x-c) + a_1\frac{(x-c)^2}{2} + a_2\frac{(x-c)^3}{3} + \cdots$$

The *radius of convergence* of the series obtained by differentiating or integrating a power series is the same as that of the original power series. The *interval of convergence*, however, may differ as a result of the behavior at the endpoints.

Example

Consider a function given by $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n} = x + \frac{x^2}{2} + \frac{x^3}{3} + \dots$

Find the intervals of convergence for each of the following.

a. $\int f(x)dx$

b. $f(x)$

c. $f'(x)$

Practice

Find the intervals of convergence for each of the following.

a. $\int f(x)dx$ b. $f(x)$ c. $f'(x)$

1 $\displaystyle\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n$

2 $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^{n+1}(x-1)^{n+1}}{n+1}$