How are parametric equations used to model motion?

Quick Check

1991 BC1

A particle moves on the $x-$axis so that its velocity any time $t \geq 0$ is given by $v(t) = 12t^2 - 36t + 15$. At $t = 1$, the particle is at the origin.

a. Find the position $x(t)$ of the particle at any time $t \geq 0$.

b. Find all values of t for which the particle is at rest.

c. Find the maximum velocity of the particle for $0 \leq t \leq 2$.

d. Find the total distance travelled by the particle from $t = 0$ to $t = 2$.

e. Find the average velocity of the particle over the interval $0 \leq t \leq 2$.

Imagine an ant walking on a xy-plane?

You stand there with a watch observing and recording its position for 5 minutes.

How would you represent the position in the $xy-$plane as a function of time?

Definition of a Plane Curve

If f and g are continuous functions of t on an interval I, then the equations $x = f(t)$ and $y = g(t)$ are called parametric equations and t is called the parameter.

Sketch the curve described by the parametic equations $x(t) = t^2 - 4$ and $y(t) = \dfrac{t}{2}$ where $-2 \leq t \leq 3$

draw graph. show direction.

t	-2	-1	0	1	2	3
x						
y						

Eliminating the parameter

Parametric equations → Solve for t in one equation. → Substitute into second equation. → Rectangular equation

$x = t^2 - 4$

$y = t/2$

$t = 2y$

$x = (2y)^2 - 4$

$x = 4y^2 - 4$

Sketch the curve represented by the equations

$$x = \frac{1}{\sqrt{t+1}} \text{ and } y = \frac{t}{t+1}, \quad t > -1$$

by eliminating the parameter and adjusting the domain of the resulting equation.

Using trigonometry to eliminate a parameter

Sketch the curve represented by
$x = 3\cos\theta$ and $y = 4\sin\theta, \quad 0 \leq \theta \leq 2\pi$
by eliminating the parameter and finding the corresponding rectangular equation.

Tangent to a parametric curve

If a smooth curve is given by the equations $x = f(t)$ and $y = g(t)$, then the slope of the curve at (x, y) is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \qquad \text{where} \quad \frac{dx}{dt} \neq 0$$

Example:

Find the slope of the curve given by $x = \sin t$ and $y = \cos t$.

Higher order derivatives

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{dx/dt}$$

2nd Derivative

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left[\frac{d^2y}{dx^2}\right] = \frac{\frac{d}{dt}\left[\frac{d^2y}{dx^2}\right]}{dx/dt}$$

3rd Derivative

Finding slope and concavity

For the curve given by

$$x = \sqrt{t} \text{ and } y = \frac{1}{4}(t^2 - 4), \qquad t \geq 4$$

find the slope and concavity at the point $(2, 3)$

Arc Length in Parametric Form

A smooth, non-intersecting curve on an interval $[a, b]$ has arclength

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Example:

Find the arclength of the curve given by $x = t^2$ and $y = 2t$ on the interval $0 \leq t \leq 2$.

Motion

$\langle x(t), y(t)\rangle$ is the position vector at any time t.

$\langle x'(t), y'(t)\rangle$ is the velocity vector.

$\langle x''(t), y''(t)\rangle$ is the acceleration vector.

$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$ is the speed or magnitude of the velocity vector.

$\int\limits_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$ is the length of the arc of the curve and represents the total distance travelled by the particle on the interval.

Practice

1️⃣ A particle moves in the $xy-$plane so that at any time t, the position is given by $x(t) = t^3 + 4t^2$ and $y(t) = t^4 - t^3$.

a. Find the velocity vector when $t = 1$.

b. What is the speed at $t = 1$.

c. Find the acceleration vector when $t = 2$.

Practice

2️⃣ An object moving along a curve in $xy-$plane has position $\langle x(t), y(t)\rangle$ at time t with $\dfrac{dx}{dt} = \sin(t^3)$, $\dfrac{dy}{dt} = \cos(t^2)$. At time $t = 2$, the object is at the postion $(1, 4)$.

a. Find the acceleration vector for the particle at $t = 2$.

b. Write the equation of the tangent line to the curve at the point where $t = 2$.

c. Find the speed of the vector at $t = 2$.

d. Find the position of the particle at time $t = 1$.