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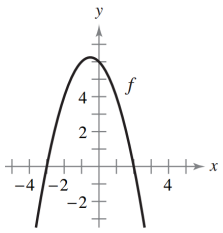
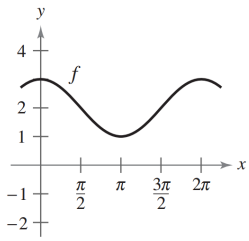
Lesson 19: How do we use implicit differentiation to find the derivative of the function?

Q: **Sketch the graphs of f' and f''.**





Most of the functions we have dealt with so far have been described by an equation of the form $y = f(x)$ that express y explicitly in terms of the variable x. We have learned rules for differentiating functions defined in this way. Another situation occurs when we encounter equations like

$$x^3 + y^3 - 9xy = 0$$

$$y^2 - x = 0$$

or

$$x^2 + y^2 - 25 = 0$$

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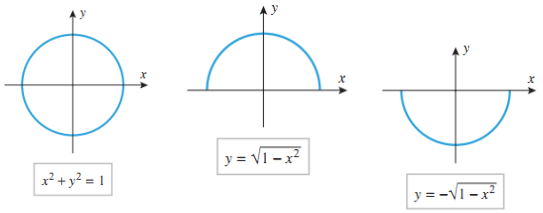
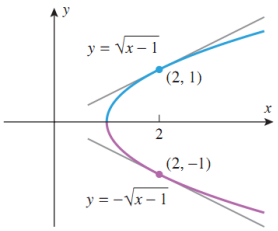


DEFINITION We will say that a given equation in x and y defines the function f ***implicitly*** if the graph of $y = f(x)$ coincides with a portion of the graph of the equation.



$$y^2 - x + 1 = 0$$

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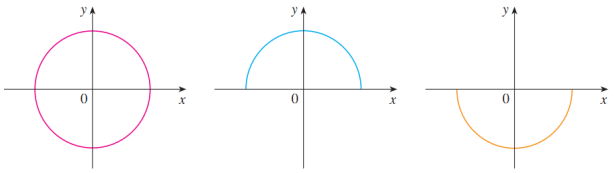

(a) $x^2 + y^2 = 25$ (b) $f(x) = \sqrt{25 - x^2}$ (c) $g(x) = -\sqrt{25 - x^2}$

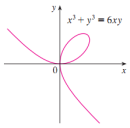

FIGURE 2 The folium of Descartes

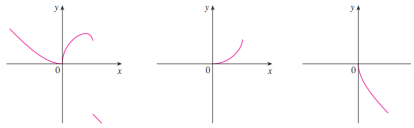

FIGURE 3 Graphs of three functions defined by the folium of Descartes

$$x^3 + [f(x)]^3 = 6xf(x)$$

EXAMPLE 1 Differentiating with Respect to x

a. $\dfrac{d}{dx}[x^3] = 3x^2$ Variables agree: use Simple Power Rule.

Variables agree

b. $\dfrac{d}{dx}[\overbrace{y^3}^{u^n}] = \overbrace{3y^2}^{nu^{n-1}}\overbrace{\dfrac{dy}{dx}}^{u'}$ Variables disagree: use Chain Rule.

Variables disagree

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c. $\frac{d}{dx}[x + 3y] = 1 + 3\frac{dy}{dx}$ Chain Rule: $\frac{d}{dx}[3y] = 3y'$

d. $\frac{d}{dx}[xy^2] = x\frac{d}{dx}[y^2] + y^2\frac{d}{dx}[x]$ Product Rule

$= x\left(2y\frac{dy}{dx}\right) + y^2(1)$ Chain Rule

$= 2xy\frac{dy}{dx} + y^2$ Simplify.

Guidelines for Implicit Differentiation

1. Differentiate both sides of the equation *with respect to x*.
2. Collect all terms involving dy/dx on the left side of the equation and move all other terms to the right side of the equation.
3. Factor dy/dx out of the left side of the equation.
4. Solve for dy/dx.

Example 1:

(a) If $x^2 + y^2 = 25$, find $\frac{dy}{dx}$.

(b) Find an equation of the tangent to the circle $x^2 + y^2 = 25$ at the point $(3, 4)$.

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Try it

Use implicit differentiation to find the derivative of the following function.

$$x^2 + y^2 = 36$$

Example 2

Find dy/dx given that $y^3 + y^2 - 5y - x^2 = -4$.

Try it

Use implicit differentiation to find the derivative of the following function.

$$x^3 - xy + y^2 = 4$$

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Example 3

Use implicit differentiation to find dy/dx if $5y^2 + \sin y = x^2$

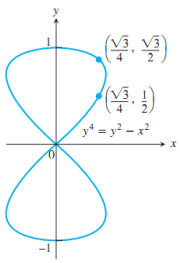
Example 4 Find $\dfrac{dy}{dx}$ $\sin x + 2\cos 2y = 1$

Example 5:

Use implicit differentiation to find d^2y/dx^2 if $4x^2 - 2y^2 = 9$.

1) Find the slopes of the tangent lines to the curve $y^2 - x + 1 = 0$ at the points $(2, -1)$ and $(2, 1)$.

2) **The eight curve** Find the slopes of the curve $y^4 = y^2 - x^2$ at the two points shown here.



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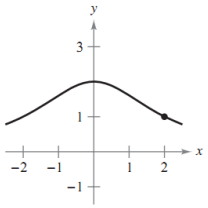
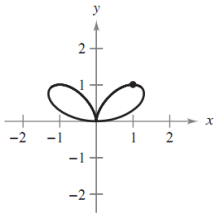
Find the equation of the tangent line to the graph at the given point

1) Witch of Agnesi:
$(x^2 + 4)y = 8$
Point: (2, 1)



2) Bifolium:
$(x^2 + y^2)^2 = 4x^2y$
Point: (1, 1)



The *normal line* at a point is perpendicular to the tangent line at the point.

1. Find an equation for the line perpendicular to the tangent to the curve $y = x^3 - 4x + 1$ at the point (2,1).

2. Find equations for the tangent line and normal line to the circle at the given points.

$$x^2 + y^2 = 25$$
$$(4, 3)$$

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find d^2y/dx^2 in terms of x and y

45. $x^2 + y^2 = 36$

47. $x^2 - y^2 = 16$

49. $y^2 = x^3$

find dy/dx by implicit differentiation

1. $x^2 + y^2 = 36$

3. $x^{1/2} + y^{1/2} = 9$

5. $x^3 - xy + y^2 = 4$

7. $x^3y^3 - y = x$

9. $x^3 - 3x^2y + 2xy^2 = 12$

11. $\sin x + 2\cos 2y = 1$